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RIGIDITY AND TOPOLOGICAL CONJUGATES OF TOPOLOGICALLY TAME KLEINIAN GROUPS

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ABSTRACT. Minsky proved that two Kleinian groups G_1 and G_2 are quasi-conformally conjugate if they are freely indecomposable, the injectivity radii at all points of $\mathbf{H}^3/G_1$, $\mathbf{H}^3/G_2$ are bounded below by a positive constant, and there is a homeomorphism h from a topological core of $\mathbf{H}^3/G_1$ to that of $\mathbf{H}^3/G_2$ such that h and h^{-1} map ending laminations to ending laminations. We generalize this theorem to the case when G_1 and G_2 are topologically tame but may be freely decomposable under the same assumption on the injectivity radii. As an application, we prove that if a Kleinian group is topologically conjugate to another Kleinian group which is topologically tame and not a free group, and both Kleinian groups satisfy the assumption on the injectivity radii as above, then they are quasi-conformally conjugate.

1. Introduction

One of the most important problems in the Kleinian group theory is to classify Kleinian groups up to conformal conjugation. Thanks to Ahlfors, Bers, Kra, Marden, and Sullivan among others, the quasi-conformal deformation space of a given finitely generated Kleinian group is well understood. Thus the problem is to classify Kleinian groups up to quasi-conformal deformations. When two Kleinian groups Γ_1 and Γ_2 are geometrically finite, Marden proved in [15] that Γ_2 is a quasi-conformal deformation of Γ_1 if and only if there is a homeomorphism from $\mathbf{H}^3/\Gamma_1$ to $\mathbf{H}^3/\Gamma_2$ preserving the cusps. We are interested in finding information, for instance some kind of invariant, for judging that one Kleinian group is a quasi-conformal deformation of another in general even when they are geometrically infinite. One candidate for this is an invariant determined by ending laminations.

Thurston conjectured that for "geometrically tame Kleinian groups" Γ_1 and Γ_2, if there is a homeomorphism $h : \mathbf{H}^3/\Gamma_1 \to \mathbf{H}^3/\Gamma_2$ preserving cusps which maps geometrically finite ends of $\mathbf{H}^3/\Gamma_1$ to those of $\mathbf{H}^3/\Gamma_2$ and ending laminations of $\mathbf{H}^3/\Gamma_1$ to those of $\mathbf{H}^3/\Gamma_2$, then there is a quasi-conformal homeomorphism $\omega : S^2 \to S^2$ such that $\omega\gamma\omega^{-1} = h_\#(\gamma)$ for any $\gamma \in \Gamma_1$. Minsky proved in [20] that this conjecture is true if Γ_1 and Γ_2 are freely indecomposable and there is a positive lower bound for the injectivity radii at all points of $\mathbf{H}^3/\Gamma_1$ and $\mathbf{H}^3/\Gamma_2$. One of our main theorems of this paper is a generalization of this ending lamination theorem of Minsky's to the case when Γ_1 and Γ_2 are topologically tame but may be freely decomposable under the same assumption on the injectivity radii (Theorem 4.1).

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Using his ending lamination theorem, Minsky also proved that for two Kleinian groups Γ_1 and Γ_2 isomorphic to a closed surface group with the assumption on the injectivity radii, if they are topologically conjugate, i.e., if there is a homeomorphism $f : S^2 \to S^2$ such that $f\Gamma_1 f^{-1} = \Gamma_2$, then they are quasi-conformally conjugate, i.e., there is a quasi-conformal homeomorphism $\omega : S^2 \to S^2$ such that $\omega\gamma\omega^{-1} = f\gamma f^{-1}$ for any $\gamma \in \Gamma_1$. We generalized this result to the case when Γ_1 and Γ_2 are freely indecomposable with the same assumption on the injectivity radii in Ohshika [26]. We shall prove that this result can be generalized further to the case when Γ_1 is topologically tame but may be freely decomposable with the same assumption on the injectivity radii for $\mathbf{H}^3/\Gamma_1$ and $\mathbf{H}^3/\Gamma_2$ in this paper. In the case when Γ_1 is a free group, we need to assume further that Γ_2 is also topologically tame (Theorem 5.1).

Minsky's ending lamination theorem was proved by using the result in [19] which states that for freely indecomposable Kleinian groups with the assumption on the injectivity radii, the hyperbolic structures on pleated surfaces going to an end are approximated by a Teichmüller geodesic ray. We shall generalize this result to the case when Kleinian groups are topologically tame first to prove Theorem 4.1 (Theorem 3.1). For proving Theorem 3.1, we shall use the technique of Canary in [3] to construct a finite branched cover of $\mathbf{H}^3/\Gamma$ for a freely decomposable topologically tame Kleinian group Γ, which is homeomorphic to the interior of a boundary-irreducible 3-manifold and admits a negatively curved metric such that the covering projection is locally isometric outside a tubular neighbourhood of the branching locus.

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2. Preliminaries

2.1. Generalities on Kleinian groups. Kleinian groups are discrete subgroups of the Lie group $PSL_2\mathbf{C}$. From now on, we always assume that Kleinian groups are finitely generated, and torsion free. An element of $PSL_2\mathbf{C}$ which is conjugate to an element represented by a matrix of the form $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is called a parabolic element. We shall only deal with Kleinian groups without parabolic elements. As $PSL_2\mathbf{C}$ is the group of orientation-preserving isometries of the hyperbolic 3-space $\mathbf{H}^3$, for a Kleinian group Γ, we can consider the complete hyperbolic 3-manifold $\mathbf{H}^3/\Gamma$. The action of $PSL_2\mathbf{C}$ is the natural extension of the conformal action on S^2 as linear fractional transformations when we regard S^2 as the sphere at infinity S^2_∞ of $\mathbf{H}^3$.

For a Kleinian group Γ, the closure of the set consisting of fixed points in S^2_∞ for elements of Γ is called the limit set of Γ and denoted by Λ_Γ. Its complement is called the region of discontinuity of Γ and denoted by Ω_Γ. By Ahlfors' finiteness theorem, Ω_Γ/Γ is a possibly disconnected Riemann surface of finite type, since we assumed that Γ is finitely generated and torsion free.

The convex core of $\mathbf{H}^3/\Gamma$ is the minimal convex submanifold of $\mathbf{H}^3/\Gamma$ which is a deformation retract. (The convex core is a 3-manifold unless Γ is a Fuchsian group.) The convex core of $\mathbf{H}^3/\Gamma$ is equal to the quotient of the convex hull of Λ_Γ in $\mathbf{H}^3$ by Γ. A Kleinian group Γ is said to be geometrically finite if and only if the convex core of $\mathbf{H}^3/\Gamma$ has finite volume. (As we assumed that Γ has no parabolic elements, this means that the convex core is compact.) Otherwise Γ is said to be geometrically infinite. Any closed geodesic in $\mathbf{H}^3/\Gamma$ is contained in its convex core.

Hence if Γ is geometrically finite, each end of $\mathbf{H}^3/\Gamma$ has a neighbourhood which contains no closed geodesics. More generally, even for a Kleinian group Γ which may not be geometrically finite, an end e of $\mathbf{H}^3/\Gamma$ is said to be geometrically finite if it has a neighbourhood which contains no closed geodesics. (Otherwise the end is said to be geometrically infinite.) If an end of $\mathbf{H}^3/\Gamma$ is geometrically finite, it corresponds naturally to a component of Ω_Γ/Γ.

By the core theorem of Peter Scott [29], there always exists a compact 3-submanifold of $\mathbf{H}^3/\Gamma$ such that the inclusion is a homotopy equivalence since Γ is finitely generated. We call such a submanifold a topological core of $\mathbf{H}^3/\Gamma$. When Γ is geometrically finite (but not a Fuchsian group), the convex core of $\mathbf{H}^3/\Gamma$ is also a topological core. If C_1 and C_2 are topological cores of $\mathbf{H}^3/\Gamma$, then there is a homeomorphism $h : C_1 \to C_2$ such that $\iota_{2\#}h_\#(\gamma) = \iota_{1\#}(\gamma)$ for any $\gamma \in \pi_1(C_1)$ if we choose basepoints appropriately, where ι_1 and ι_2 are the inclusions of C_1 and C_2 respectively (McCullough-Miller-Swarup [21]). Each component of the complement of a topological core contains exactly one end. On the other hand, no two boundary components of a topological core are contained in the closure of a component of the complement. Hence there is a one-to-one correspondence between boundary components of a topological core and ends. We say that an end faces a boundary component of a core when the end corresponds to the boundary component by the correspondence above. We also say that a component of the complement of a topological core faces a boundary component of the topological core when the latter is in the closure of the former.

A Kleinian group Γ is said to be freely indecomposable when Γ does not have a non-trivial decomposition into a free product. (Otherwise, Γ is said to be freely decomposable.) If Γ is freely indecomposable, its topological core is boundary-irreducible and unique up to isotopy.

2.2. **3-manifold topology.**

In this subsection, we shall briefly review terminology in 3-manifold topology which will be used in this paper. For further details, the readers should refer to Jaco [11].

A 3-manifold is said to be irreducible when any embedded sphere in M bounds a 3-ball. Let M be a 3-manifold, and let S be an embedded surface in M. A compression disc for S is an embedded disc D in M such that $D \cap S = \partial D$ and ∂D is not null-homotopic on S. An embedded surface S is said to be compressible when there is a compression disc for S; otherwise S is incompressible. If S is two-sided, S is incompressible if and only if the inclusion induces a monomorphism from the fundamental group of S to that of M.

A compact 3-manifold with boundary is said to be boundary-irreducible when each boundary component is incompressible.

A compression body is a compact 3-manifold which can be constructed in the following fashion. Let S be a possibly disconnected closed surface without sphere component. Consider the product $S \times I$ and attach finitely many 1-handles to $S \times \{1\}$ so that at least one 1-handle is attached to each component of $S \times \{1\}$. Note that a compression body has only one compressible boundary component, and the other components are incompressible.

Let M be a compact irreducible 3-manifold with boundary. Then there is a compact 3-submanifold V in M homeomorphic to a compression body such that $\partial M \cap V$ is the union of compressible boundary components of ∂M, and the closure of $M - V$ is a boundary-irreducible 3-manifold. (See Bonahon [1].) Such a submanifold is called a characteristic compression body of M. A characteristic compression body is unique up to isotopy.

An embedded surface S in a 3-manifold is said to be properly embedded when $S \cap \partial M = \partial S$. A boundary-compression (semi-)disc for a properly embedded surface S is an embedded disc Δ such that both $\Delta \cap S$ and $\Delta \cap \partial M$ are arcs in $\partial\Delta$, and $\partial\Delta \cap S$ is not homotopic to an arc in ∂S preserving endpoints. A properly embedded surface S is said to be boundary-compressible if there is a boundary-compression disc for S; otherwise S is boundary-incompressible.

2.3. **Geodesic laminations and pleated surfaces.** Let S be a complete hyperbolic surface. A geodesic lamination on S is a closed subset of S consisting of disjoint simple geodesics. A measured lamination on S is a geodesic lamination with a transverse invariant measure, i.e., a positive Borel measure on arcs intersecting the lamination transversely which is invariant under isotopies preserving the leaves of the lamination. For a measured lamination λ, we call the subset of the underlying geodesic laminations consisting of points x such that there is no transverse arc containing x in the interior whose measure is 0, the support of λ. Obviously the support is again a geodesic lamination. The set of measured laminations on S, denoted by $\mathcal{ML}(S)$, is given a topology induced from the measures. With respect to this topology, the space of measured laminations is homeomorphic to the Teichmüller space of S. The projective lamination space $\mathcal{PL}(S)$ is the quotient space of the space of non-empty measured laminations obtained by identifying scalar multiples. Elements of $\mathcal{PL}(S)$ are called projective laminations. We define the support of a projective lamination similarly to that of a measured lamination. The notion of measured lamination is first introduced by Thurston [28]. Refer to Casson-Bleiler [7] for a further exposition on the theory of geodesic lamination.

A length-preserving map $f : S \to M$ from a complete hyperbolic surface to a complete hyperbolic 3-manifold is called a pleated surface when for any point $x \in S$, there is a geodesic arc containing x in the interior which is mapped geodesically by f. A geodesic lamination λ is said to be realized by a pleated surface f when $f|\lambda$ is totally geodesic. A measured lamination or a projective lamination is said to be realized by a pleated surface when its support is realized by the pleated surface. Refer to Thurston [28] and Canary-Epstein-Green [5] for further explanation.

2.4. **Tameness of Kleinian groups.** A Kleinian group Γ and $\mathbf{H}^3/\Gamma$ are said to be topologically tame if $\mathbf{H}^3/\Gamma$ is homeomorphic to the interior of a compact 3-manifold. Marden conjectured in [15] that all finitely generated Kleinian groups are topologically tame; this is still an open problem. Bonahon proved that if Γ is freely indecomposable, it is topologically tame ([2]). This was proved by showing that if Γ is freely indecomposable, then it is geometrically tame in the following sense. Let Γ be a freely indecomposable Kleinian group and let C be a topological core of $\mathbf{H}^3/\Gamma$. Suppose that an end e of $\mathbf{H}^3/\Gamma$ is geometrically infinite and faces a boundary component S of C. In this situation, Bonahon proved that there exists a sequence of simple closed curves $\{\gamma_i\}$ on S such that the closed geodesic γ_i^* homotopic to γ_i in $\mathbf{H}^3/\Gamma$ tends to the end e as $i \to \infty$. If such a sequence exists (for an end e facing an incompressible boundary component S of a topological core), the end e is said to be geometrically tame. In other words, Bonahon proved that if Γ is freely indecomposable, every geometrically infinite end of $\mathbf{H}^3/\Gamma$ is geometrically tame. (The group Γ itself is said to be geometrically tame then.) Thurston proved in [28] that if the end e, facing an incompressible boundary components of a topological core, is geometrically tame, then it has a neighbourhood homeomorphic to $S \times \mathbf{R}$. In particular, if Γ is geometrically tame, then it is also topologically tame.

Let us now turn to the case when Γ is freely decomposable. Let C be a topological core of $\mathbf{H}^3/\Gamma$. Let S be a boundary component of C. Let $\mathcal{C}(S)$ be the subset of the measured lamination space $\mathcal{ML}(S)$ consisting of disjoint unions of weighted simple closed curves which bound compression discs. When Γ is not isomorphic to a free product of two closed surface groups, following Otal [27], we define the Masur domain of S to be

$$\mathcal{M}(S) = \{\lambda \in \mathcal{ML}(S) | i(\lambda, \mu) \neq 0 \text{ for every } \mu \in \overline{\mathcal{C}}(S)\},$$

where $\overline{\mathcal{C}}(S)$ denotes the closure of $\mathcal{C}(S)$. In the case when Γ is isomorphic to a free product of two closed surface groups, we define the Masur domain to be

$$\begin{aligned}\mathcal{M}(S) =\{&\lambda \in \mathcal{ML}(S) | i(\lambda, \mu) \neq 0 \text{ for every } \mu \text{ such that}\\ &\text{there is } \nu \in \overline{\mathcal{C}}(S) \text{ with } i(\mu, \nu) = 0\}.\end{aligned}$$

As $\mathcal{M}(S)$ is preserved under scalar multiplication, we can define its projectivization $\mathcal{PM}(S) \subset \mathcal{PL}(S)$ and call it the projectivized Masur domain. Note that if S is incompressible then $\mathcal{M}(S) = \mathcal{ML}(S)$.

In the case when Γ is possibly freely decomposable as above, the end e of $\mathbf{H}^3/\Gamma$ facing a boundary component S of a topological core C is said to be "geometrically tame" (we use this term, which may be confusing, only in this section) if there is a sequence of simple closed curves $\{\gamma_i\}$ on S whose projective classes converge to a projective lamination in $\mathcal{PM}(S)$ such that the closed geodesic γ_i^* homotopic to γ_i in $\mathbf{H}^3/\Gamma$ tends to the end e as $i \to \infty$. Note that if S is incompressible, this definition coincides with the previous one. Unless Γ is a free group, if e is a geometrically tame end facing S, then there is a neighbourhood of e which is homeomorphic to $S \times \mathbf{R}$. (See Ohshika [24].)

Let e be a geometrically tame end of $\mathbf{H}^3/\Gamma$ which has a neighbourhood E homeomorphic to $S \times \mathbf{R}$ (this is always the case unless the Kleinian group is free). Let C be a topological core of $\mathbf{H}^3/\Gamma$ such that E is a component of the complement of C. A measured lamination λ in the Masur domain or its projective class $[\lambda]$ is said to be an ending lamination in the stronger sense if there is a sequence of simple closed curves $\{\gamma_i\}$ on S, whose projective classes converge to $[\lambda]$ in $\mathcal{PL}(S)$ and such that there is a closed geodesic γ_i^* homotopic to γ_i in E which tends to e as $i \to \infty$. As was shown in § 10 of Canary [3], for any geometrically infinite end which has a neighbourhood homeomorphic to the product of a closed surface and an open interval, an ending lamination in the stronger sense exists and is unique up to transverse measures. In the case when the group is not isomorphic to a free group the following lemma holds; it enables us to define ending laminations without referring to homotopies in E.

Lemma 2.1. *Suppose that Γ is not isomorphic to a free group. Let λ be a measured lamination on S contained in the Masur domain $\mathcal{M}(S)$, and suppose that there exists a sequence of weighted simple closed curves $\{w_i\gamma_i\}$ converging to λ such that the closed geodesic γ_i^* homotopic to γ_i in $\mathbf{H}^3/\Gamma$ tends to the end e as $i \to \infty$. Then there exits a homeomorphism $h : S \to S$ acting on $\pi_1(C)$ by an inner automorphism such that $h(\lambda)$ is an ending lamination for e.*

Proof. By Proposition 4.12 in Ohshika [24], there are a topological core C' of $\mathbf{H}^3/\Gamma$ such that the component E of its complement containing e is homeomorphic to

$S \times \mathbf{R}$, and a pleated surface $g_i : S \to \mathbf{H}^3/\Gamma - C'$ containing γ_i^* in its image which is homotopic in $\mathbf{H}^3/\Gamma - C'$ to the boundary component S' facing e for sufficiently large i. Since Γ is not a free group, S' is not null-homologous, and neither is g_i. Moreover, as is shown in the proof of Proposition 4.12 in [24], for any given compact set K in $\mathbf{H}^3/\Gamma$, there exists i_0 such that if $i \geq i_0$, then the pleated surface g_i is homotopic in $\mathbf{H}^3/\Gamma - K$ to an embedding of S. An embedding of S which is not null-homologous must be isotopic to a homeomorphism onto S in $E \cup S$. (This can be shown by compressing the embedding to get an incompressible surface.) Hence for sufficiently large i, the pleated surface g_i is homotopic to an embedding onto S in $E \cup S$. It follows that there is a simple closed curve $\overline{\gamma}_i$ on S which is homotopic to γ_i^* in $E \cup S$, and that there is a homeomorphism $h_i : S \to S$ homotopic to the identity in $\mathbf{H}^3/\Gamma$ which maps γ_i to $\overline{\gamma}_i$. By the definition of ending lamination in the stronger sense, there is an ending lamination in the stronger sense $\overline{\lambda}$ for e on S such that $[\overline{\gamma}_i]$ converges to $[\overline{\lambda}]$ in $\mathcal{PL}(S)$. By assumption and Canary's result mentioned in the definition of ending lamination in the stronger sense above, both λ and $\overline{\lambda}$ are contained in the Masur domain. Since the group of auto-homeomorphisms of S acting on $\pi_1(C)$ by inner automorphisms acts on the projectivized Masur domain properly discontinuously (see Otal [27]), we can assume that h_i does not depend on i by taking a subsequence. This implies that there is a homeomorphism $h : S \to S$ acting on $\pi_1(C)$ by an inner automorphism such that $h(\lambda) = \overline{\lambda}$. This completes the proof of the lemma. □

When we consider two isomorphic Kleinian groups G_1 and G_2 such that the topological cores of $\mathbf{H}^3/G_1$ and $\mathbf{H}^3/G_2$ are homeomorphic, an isomorphism between G_1 and G_2 determines an isotopy class of homeomorphisms of cores only up to auto-homeomorphisms of a core acting on the fundamental group by inner automorphisms. Thus it will be convenient to define ending laminations so that their images by such auto-homeomorphisms are also ending laminations.

Definition 2.1. A measured lamination λ on S is said to be an ending lamination of the end e if there is a homeomorphism of C acting on $\pi_1(C)$ by an inner automorphism such that $h(\lambda)$ is an ending lamination in the stronger sense.

In the case when Γ is not isomorphic to a free group, by the lemma above, this is equivalent to saying that a measured lamination λ in $\mathcal{M}(S)$ is an ending lamination if there exists a sequence of weighted simple closed curves $\{w_i\gamma_i\}$ converging to λ such that the closed geodesic γ_i^* homotopic to γ_i in $\mathbf{H}^3/\Gamma$ tends to the end e as $i \to \infty$. In the case when Γ is freely indecomposable, the two notions, ending lamination and ending lamination in the stronger sense, coincide, and they are unique modulo transverse measures. Ending laminations can be defined for any negatively curved manifold each of whose ends has a neighbourhood where the sectional curvature is constantly -1.

2.5. **Canary's construction.** By virtue of Bonahon's theory, freely indecomposable Kleinian groups are understood better than freely decomposable Kleinian groups. Canary's work in [3] made it possible to apply Bonahon's theory to topologically tame Kleinian groups which may be freely decomposable. Let us explain his result briefly. Let Γ be a topologically tame Kleinian group. Then $\mathbf{H}^3/\Gamma$ can be regarded as the interior of a compact 3-manifold M. Let γ be a collection of closed geodesics in $\mathbf{H}^3/\Gamma$ which is null-homologous after giving appropriate orientations to the components, and intersects all compression discs of M essentially. Even when

γ itself is not simple, by perturbing the metric of $\mathbf{H}^3/\Gamma$ in small balls keeping the curvature pinched, we can assume that γ is simple. Take a 3-fold cyclic branched cover $\overline{\tilde{M}}$ branching along γ. Then $\overline{\tilde{M}}$ is boundary-irreducible (i.e., $\partial\overline{\tilde{M}}$ is incompressible), and acylindrical (i.e., there are no essential annuli properly embedded in $\overline{\tilde{M}}$). The interior of $\overline{\tilde{M}}$, which we denote by $\tilde{M}$, admits a Riemannian metric with pinched negative curvature such that the covering projection is isometric outside a tubular neighbourhood $\mathcal{N}(\tilde{\gamma})$ of the lift $\tilde{\gamma}$ of the branching locus γ. In particular, each end of $\mathbf{H}^3/\Gamma$ is lifted to ends of $\tilde{M}$, to each of which we can apply Bonahon's theory. Using this method, Canary proved that if Γ is topologically tame, then each end of $\mathbf{H}^3/\Gamma$ is geometrically tame in the sense defined in the last subsection, has ending laminations in the stronger sense, and there is a family of hyperbolic simplicial surfaces filling a neighbourhood of each end (which corresponds to $S \times \{t\}$ up to homotopy for a neighbourhood homeomorphic to $S \times \mathbf{R}$).

2.6. **Quasi-conformal maps.** A homeomorphism $\omega : S^2 \to S^2$ is said to be quasi-conformal if it has locally L^p-distributional derivatives $\omega_z, \omega_{\overline{z}}$ almost everywhere for $p \geq 1$, and $\|\omega_{\overline{z}}/\omega_z\|_\infty < 1$. Conversely, for a given measurable complex function $\mu : S^2 \to \mathbf{C}$ with $\|\mu\|_\infty < 1$, which is called a Beltrami coefficient, the Beltrami equation $\omega_{\overline{z}} = \mu\omega_z$ (almost everywhere) can be solved and gives a quasi-conformal homeomorphism $\omega : S^2 \to S^2$. (Refer for example to Lehto [14] for further details on quasi-conformal maps.)

For a Kleinian group Γ, if μ satisfies the condition that $\mu(\gamma z)\overline{\gamma'(z)}/\gamma'(z) = \mu(z)$ for any $\gamma \in \Gamma$ and $z \in S^2$, and ω is a quasi-conformal homeomorphism which is a solution of the equation $\omega_{\overline{z}} = \mu\omega_z$, then $\omega\Gamma\omega^{-1}$ is again a Kleinian group. In this situation, we call $\omega\Gamma\omega^{-1}$ a quasi-conformal deformation of Γ, and we also say that $\omega\Gamma\omega^{-1}$ and Γ are quasi-conformally conjugate.

Similarly, we can consider quasi-conformal deformations of Fuchsian groups by conjugating Fuchsian groups by quasi-conformal homeomorphisms from the upper half plane to itself. Let S be a closed surface of genus greater than 1. The space of marked conformal structures (or hyperbolic structures) modulo conformal homeomorphisms (or isometries) isotopic to the identity is called the Teichmüller space of S and denoted by $\mathcal{T}(S)$. A point σ of $\mathcal{T}(S)$ corresponds to a Fuchsian group Γ_σ with an isomorphism $\pi_1(S) \to \Gamma_\sigma$. This correspondence is unique up to conjugation in $PSL_2\mathbf{R}$. If we fix $\sigma \in \mathcal{T}(S)$, then for any point $\sigma' \in \mathcal{T}(S)$, a Fuchsian group $\Gamma_{\sigma'}$ is obtained as a quasi-conformal deformation of Γ_σ. In other words, for any two points $\sigma, \sigma' \in \mathcal{T}(S)$, there is a quasi-conformal homeomorphism from (S, σ) to (S, σ') isotopic to the identity. (A quasi-conformal homeomorphism from (S, σ) to (S, σ') can be defined similarly to the case of the upper half space by using local coordinates.) For a quasi-conformal homeomorphism $f : (S, \sigma) \to (S, \sigma')$, the maximal dilatation of f is defined to be $K_f = \sup_{z \in (S,\sigma)} \frac{1+|\mu(z)|}{1-|\mu(z)|}$, where $\mu = f_{\overline{z}}/f_z$. For two points σ, σ' in $\mathcal{T}(S)$, the Teichmüller distance between σ and σ' is defined to be the infinimum of $\log K_f$, where f ranges over all quasi-conformal homeomorphisms from (S, σ) to (S, σ') isotopic to the identity, and we denote it by $d_T(\sigma, \sigma')$. There is a quasi-conformal homeomorphism from (S, σ) to (S, σ') isotopic to the identity whose logarithm of maximal dilatation is equal to $d_T(\sigma, \sigma')$, which is called a Teichmüller map. If f is a Teichmüller map from (S, σ) to (S, σ'), then its Beltrami coefficient has the form $k\overline{\phi}/|\phi|$ for some holomorphic quadratic differential ϕ. A quadratic differential ϕ determines two measured foliations, the horizontal foliation Φ_h whose tangent vector field v_h is defined by $\phi(z)dz^2(v_h) > 0$, and the vertical foliation Φ_v whose tangent vector field v_v is defined by $\phi(z)dz^2(v_v) < 0$. They form horizontal and vertical directions of a Euclidean structure with isolated singularities defined by $|\phi(z)|^{1/2}|dz|$. A Teichmüller map whose Beltrami coefficient is $k\overline{\phi}/|\phi|$ is regarded as stretching a conformal structure along the horizontal foliation Φ_h of ϕ. By considering the one-parameter family of Beltrami coefficients $\mu_t = t\overline{\phi}/|\phi|$ $(0 \leq t < 1)$ on (S, σ), we get a one-parameter family of quasi-conformal homeomorphisms $f_t : (S, \sigma) \to (S, \sigma_t)$, which gives rise to a Teichmüller geodesic ray $\{\sigma_t\}$ in $\mathcal{T}(S)$, a geodesic with respect to the Teichmüller distance such that $\sigma_0 = \sigma$.

Let γ be a simple closed curve on (S, σ). The extremal length $E_\sigma(\gamma)$ of γ is defined to be $\sup_\rho \mathrm{length}_\rho(\gamma)^2/\mathrm{Area}(S, \rho)$, where ρ ranges over all metrics conformal to σ. Alternatively, $E_\sigma(\gamma)$ is equal to the $\inf_C 1/\mathrm{mod}(C)$, where C ranges over all cylinders with core curve γ and $\mathrm{mod}(C)$ denotes the modulus (the height/the perimeter of the base) of C. The extremal length is extended continuously to the space of measured foliations $\mathcal{MF}(S)$. (See Kerckhoff [13].)

For $\sigma, \sigma' \in \mathcal{T}(S)$, their Teichmüller distance is equal to $\sup_\gamma E_{\sigma'}(\gamma)/E_\sigma(\gamma)$, where γ ranges over all simple closed curves on S. The extremal length function is extended continuously to the space of measured foliations, and the supremum above is attained by the horizontal foliation associated to the holomorphic quadratic differential defining the Beltrami coefficient of the Teichmüller map (Kerckhoff [13]).

2.7. **Harmonic maps.** Let M, N be Riemannian manifolds and suppose that N has finite volume. For a differentiable map $f : N \to M$, we define its energy to be

$$\mathcal{E}(f) = \frac{1}{2}\int_N |df|^2 dv_N,$$

where dv_N is the volume form of N. A differentiable map from N to M is said to be harmonic if it is a stationary point in the space of differentiable maps from N to M for the functional $\mathcal{E}$. In this paper, we consider only maps from a compact Riemann surface S to a negatively curved 3-manifold M. As Minsky proved in [18], if we fix a homotopy class of incompressible maps from (S, σ) to M, where σ is a hyperbolic structure on S, there is a unique harmonic map in the homotopy class. Let $f : (S, \sigma) \to (M, \rho)$ be an incompressible harmonic map. Pull back the negatively curved metric ρ of M to S by f, and denote it by $f^*\rho$. We can express $f^*\rho$ using the basis $dz^2, dzd\overline{z}, d\overline{z}^2$ with respect to the complex structure on S compatible with σ. The dz^2 part of $f^*\rho$ is a quadratic differential, which is called a Hopf differential of f. (See Jost [12].) Minsky proved in [18] that if M is hyperbolic and f has a sufficiently high energy, f maps S near to the realization by a pleated surface homotopic to f of the measured lamination corresponding to the horizontal foliation associated to the Hopf differential.

2.8. **Quasi-isometries.** Let (X, d) and (Y, d') be metric spaces. A map $f : X \to Y$ is said to be (K, δ)-quasi-Lipschitz if $d'(x_1, x_2) \leq Kd(f(x_1), f(x_2)) + \delta$ for any $x_1, x_2 \in X$, and a (K, δ)-quasi-isometry if $K^{-1}d(x_1, x_2) - \delta \leq d'(f(x_1), f(x_2)) \leq Kd(x_1, x_2) + \delta$ for any $x_1, x_2 \in X$. A continuous map $f : X \to Y$, where X, Y are path-connected metric spaces, is said to be a liftable (K, δ)-quasi-isometry if f can be lifted to a (K, δ)-quasi-isometry between universal covers of X, Y. A quasi-isometry from $\mathbf{H}^3$ to itself can be extended a quasi-conformal homeomorphism from S^2_∞ to itself. (See Thurston [28].)

A sequence of Riemannian manifolds with base frames $\{(M_i, v_i)\}$ is said to converge geometrically to a Riemannian manifold with base frame (N, w) in the sense of Gromov if there exist sequences of positive real numbers $\{r_i\}$ going to ∞ and $\{K_i\}$ going to 1 as $i \to \infty$, and for each i there exists a $(K_i, 0)$-quasi-isometry ρ_i from the r_i-ball in M_i centred at the point on which v_i lies to the r_i-ball in N centred at the point on which w lies, and such that $d\rho_i(v_i) = w$. Such a quasi-isometry is called a (K_i, r_i)-approximate isometry.

3. Hyperbolic structures on pleated surfaces

In this section, we shall generalize the main theorem in Minsky [19] to a topologically tame hyperbolic 3-manifold $M = \mathbf{H}^3/G$, where G may be freely decomposable, and in which the injectivity radii at all points are bounded below by a positive constant δ_0. For that, we shall consider the interior $\tilde{M}$ of a three-fold acylindrical branched cover $\overline{\tilde{M}}$ as explained in the last section, and give it a metric with pinched negative curvature which is constantly -1 apart from a neighbourhood $\mathcal{N}(\tilde{\gamma})$ of the lift $\tilde{\gamma}$ of the branching locus. Let e be a geometrically infinite end of M which we want to study. Let C be a topological core of $\mathbf{H}^3/G$ such that each component of its complement is homeomorphic to the product of a closed surface and an open interval. (Such a core exists since we assume that G is topologically tame.) Let S be a boundary component of C which faces e. We can assume that the branching locus γ is contained in C by enlarging C if necessary. Then the core C is covered by a topological core $\tilde{C}$ of $\tilde{M}$, since we assumed each component of the complement of C is homeomorphic to the product of a closed surface and an open interval. Let $\tilde{e}$ be an end of $\tilde{M}$ which corresponds to e by the covering projection. Then there is a homeomorphic lift $\tilde{S}$ of S which is a boundary component of $\tilde{C}$ facing $\tilde{e}$.

Throughout this section, we always assume that G is topologically tame and the injectivity radii at all points of $\mathbf{H}^3/G$ are bounded below by a positive constant δ_0. Let us state our generalization of Minsky's theorem in [19] now.

Theorem 3.1. *Let $M = \mathbf{H}^3/G$ be a topologically tame hyperbolic 3-manifold in which the injectivity radius is bounded below at any point by a universal positive constant δ_0. Let e be a geometrically infinite end of M. Let C be a topological core of M such that each component of its complement is homeomorphic to the product of a closed surface and an open interval, let S be a boundary component of C facing e, and let E be the component of $M - C$ containing e. Then there exist a neighbourhood $\overline{E}$ of e homeomorphic to $S \times \mathbf{R}$ contained in $M - C$, a Teichmüller geodesic ray L, and positive constants A, B as follows.*

1. *For any pleated surface $f : S \to \overline{E}$ homotopic to the inclusion of S in $E \cup S$, the hyperbolic structure σ_f induced by f on S is within Teichmüller distance A from L.*
2. *For any point σ in L, there exist a hyperbolic structure τ within Teichmüller distance B from σ and a pleated surface $f : S \to \overline{E}$ homotopic to the inclusion in $E \cup S$ which induces the hyperbolic structure τ on S.*

Since the statement of the theorem is proved to be valid when S is incompressible by applying the main theorem of Minsky [19] to the covering of $\mathbf{H}^3/G$ associated to $\pi_1(S)$, we assume that S is compressible from now on until the end of this section. Let $\iota : S \to \tilde{M}$ be an embedding which is a lift of the inclusion of S into M, and whose image is $\tilde{S}$. To prove the theorem above, we need to consider harmonic maps in the branched cover $\tilde{M}$ which are homotopic to $\iota : S \to \tilde{M}$. The main step which we need to prove Theorem 3.1 is Proposition 3.12, which states that for two pleated surfaces far enough in a neighbourhood of the end $\tilde{e}$, the image of the harmonic map from the hyperbolic surface with the hyperbolic structure on the Teichmüller geodesic segment connecting the hyperbolic structure on the two pleated surfaces is far from the neighbourhood $\mathcal{N}(\tilde{\gamma})$ where the sectional curvature varies. Before showing that proposition, we need to prove several lemmata. First we shall show that there is a lower bound for the injectivity radii at all points in $\tilde{M}$.

Lemma 3.2. *There exists a lower bound $\epsilon_0 > 0$ for the injectivity radii at all points of $\tilde{M}$.*

Proof. We have only to bound the lengths of closed geodesics in $\tilde{M}$ from below by a positive constant. Let K be a compact submanifold in $\tilde{M}$ containing $\mathcal{N}(\tilde{\gamma})$. Fix a positive constant δ and let K' be the set consisting of points within distance δ from K. Then K' contains K and is also a compact submanifold. Hence there exists a lower bound ϵ_1 for the lengths of closed geodesics contained in K'.

Suppose that a closed geodesic γ is not contained in K'. Then there are two possibilities: the first is that γ lies outside K; the second is that γ intersects both $\mathrm{Fr}K$ and $\mathrm{Fr}K'$. In the first case, γ does not meet $\mathcal{N}(\tilde{\gamma})$; hence γ is projected to a closed geodesic in M, which implies that the length of γ is at least $2\delta_0$. In the second case, since $d(\mathrm{Fr}K, \mathrm{Fr}K') = \delta$, the length of γ is greater than δ. Thus in any case, the length of a closed geodesic is bounded below by $\min\{\delta, 2\delta_0, \epsilon_1\}$. □

Next, we shall prove the following lemma, which describes the asymptotic structure of $\tilde{M}$ near an end $\tilde{e}$ which is a lift of the end e of M facing a boundary component S of C. The lemma (and its proof) is an analogue of Thurston's proposition in [28] asserting that a geometrically tame end is topologically tame, and its proof using the technique of interpolating pleated surfaces.

Lemma 3.3. *Let $\{v_i\}$ be a sequence of base-frames on basepoints x_i in $\tilde{M}$ which tend to the end $\tilde{e}$ as $i \to \infty$. Then, up to a subsequence, $\{(\tilde{M}, v_i)\}$ converges geometrically in the sense of Gromov to a hyperbolic 3-manifold with base-frame (M_∞, v_∞) whose fundamental group is isomorphic to $\pi_1(S)$, and in which the injectivity radii at all points are bounded below by ϵ_0.*

Proof. By Gromov's convergence theorem in [10], $\{(\tilde{M}, v_i)\}$ converges geometrically to a negatively curved 3-manifold with base frame (M_∞, v_∞) after taking a subsequence, because the injectivity radius at v_i is universally bounded below by ϵ_0. Let $\rho_i : B_{r_i}(\tilde{M}, x_i) \to B_{r_i}(M_\infty, x_\infty)$ be an approximate isometry associated to this convergence. (See Canary-Epstein-Green [5].) As $\tilde{M}$ has constant curvature -1 apart from the tubular neighbourhood $\mathcal{N}(\tilde{\gamma})$, and $d(x_i, \tilde{\gamma}) \to \infty$, it is obvious that the geometric limit M_∞ has constant curvature -1. Suppose that there is an essential loop c with length less than $2\epsilon_0$ in M_∞. Then by pulling back c by the approximate isometry ρ_i for sufficiently large i, we get a loop $\rho_i^{-1}(c)$ with length less than $2\epsilon_0$. If $\rho_i^{-1}(c)$ is essential, it contradicts the fact that the injectivity radius at any point of $\tilde{M}$ is at least ϵ_0. Suppose that $\rho_i^{-1}(c)$ is inessential. Because $\rho_i^{-1}(c)$ is sufficiently far from $\tilde{\gamma}$, the loop $\rho_i^{-1}(c)$ bounds a disc D_i not intersecting $\mathcal{N}(\tilde{\gamma})$, whose diameter is less than $2\epsilon_0$, which we can construct by lifting $\rho_i^{-1}(c)$ to $\mathbf{H}^3$. For sufficiently large i, the disc D_i is contained in an r_i-ball centred at x_i in which ρ_i is defined. Then $\rho_i(D_i)$ gives a disc bounded by c, which contradicts the essentiality of c. Thus the injectivity radius at any point of M_∞ is at least ϵ_0.

Next we shall prove that $\pi_1(M_\infty) = \pi_1(S)$. Since G is topologically tame, by Corollary 10.2 in Canary [3], there exists a sequence of simple closed curves $\{\gamma_i\}$ on S whose projective classes converge in $\mathcal{PM}(S)$ such that the closed geodesics γ_i^* homotopic to γ_i in M tend to the end e. Then, as in the proof of Proposition 4.11 in [24], which generalizes Theorem 9.5.13 in Thurston [28], there exists a one-parameter family of pleated surfaces $f_t : S \to \mathbf{H}^3/G = M$, which tends to the end e as $t \to \infty$, such that f_t is continuous with respect to t apart from countably many values of t. For the one-parameter family constructed there, if f_t is discontinuous with respect t at t_0, then $f_{t_0-0} = \lim_{t\to t_0-0} f_t$ and $f_{t_0+0} = \lim_{t\to t_0+0} f_t$ differ only in the interior of an ideal quadrilateral Q in S. Both $f_{t_0-0}(Q)$ and $f_{t_0+0}(Q)$ are unions of two ideal triangles in $\tilde{M}$, and their union $f_{t_0-0}(Q) \cup f_{t_0+0}(Q)$ bounds an ideal tetrahedron.

There is a universal constant, independent of isometry types, which bounds the volumes of ideal tetrahedra. Hence the radius of a ball contained in an ideal tetrahedron is universally bounded above. This implies that there exists a bound K such that any point in the tetrahedron bounded by $f_{t_0-0}(Q) \cup f_{t_0+0}(Q)$, for any point t_0 of discontinuity, is within distance K from $f_{t_0-0}(Q) \cup f_{t_0+0}(Q)$. As is shown in Thurston [28], whose argument is also used in the proof of Proposition 4.11 in [24], we can interpolate the one-parameter family $\{f_t\}$ by filling each gap at t_0 by a homotopy from $f_{t_0-0}(Q)$ to $f_{t_0+0}(Q)$, and get a continuous proper map $F : S \times [0,\infty) \to M$ which is surjective to a neighbourhood of the end e. It follows that there exist a constant K and a neighbourhood $\overline{E}$ of e such that any point $x \in \overline{E}$ is at distance less than K from a pleated surface homotopic to the inclusion of S. By lifting this to a neighbourhood of the end $\tilde{e}$ of $\tilde{M}$ to which $\{x_i\}$ tends, we can see that for sufficiently large i, there is a pleated surface homotopic to $\iota : S \to \tilde{M}$ within distance K from x_i. Such incompressible pleated surfaces converge (as unmarked pleated surfaces) to an incompressible pleated surface in M_∞ after taking a subsequence, because the injectivity radii are bounded below. (See Thurston [31] and Canary-Epstein-Green [5].) In other words, there exist a pleated surface $f_i : S \to \tilde{M}$ homotopic to ι and a homeomorphism $h_i : S \to S$ such that such that $d(f_i(S), x_i) \leq K$ and $f_i \circ h_i$ converges to a pleated surface $f_\infty : S \to M_\infty$ as marked pleated surfaces. Since the base points x_i converge to the base point $x_\infty \in M_\infty$, we have $d(f_\infty(S), x_\infty) \leq K$. Furthermore, $\pi_1(M_\infty)$ contains $f_{\infty\#}(\pi_1(S))$ and $\rho_{i\#}^{-1}(f_{\infty\#}(\pi_1(S))) = f_{i\#}\pi_1(S)$.

Let us now prove that $\pi_1(M_\infty) = f_{\infty\#}\pi_1(S)$. Let $q : \tilde{M}_\infty = \mathbf{H}^3/\tilde{\Gamma}_\infty \to M_\infty$ be a covering associated with $f_{\infty\#}\pi_1(S)$. Since the injectivity radius at any point of M_∞ is at least ϵ_0, so is the injectivity radius at any point of $\tilde{M}_\infty$. In particular, $\tilde{\Gamma}_\infty$ has no parabolic elements. Then $\tilde{\Gamma}_\infty$ is either a quasi-Fuchsian group or a totally degenerate b-group without parabolics or a totally doubly degenerate group without parabolics. (A Kleinian group G isomorphic to a closed surface group is said to be totally degenerate if the region of discontinuity Ω_G is connected, and doubly degenerate if $\Omega_G = \emptyset$.) We shall prove that the last is the case; in other words, $\mathbf{H}^3/\tilde{\Gamma}_\infty$ has two geometrically infinite tame ends.

Lemma 3.4. *The Kleinian group $\tilde{\Gamma}_\infty$ is a doubly degenerate group without parabolics.*

Proof. Let $\tilde{E}$ be the component of the complement of the topological core $\tilde{C}$ of $\tilde{M}$ which contains $\tilde{e}$, whose frontier is $\tilde{S}$. Let $g : S \to \tilde{E}$ be a pleated surface homotopic to $\iota : S \to \tilde{S}$ in $\tilde{E} \cup \tilde{S}$. Let $\mu_0 \subset S$ be a measured lamination realized by g and let $\mu \subset \tilde{S}$ be $\iota(\mu_0)$. Let $\lambda \subset \tilde{S}$ be an ending lamination of $\tilde{e}$ and let λ_0 be $\iota^{-1}(\lambda)$. Recall that f_∞ is a limit of $\{f_i \circ h_i\}$ as marked pleated surfaces, for pleated surfaces f_i homotopic to $\iota : S \to \tilde{S}$. Therefore for sufficiently large i, the map $\rho_i^{-1} \circ f_\infty$ is homotopic in $\tilde{M}$ to the homeomorphism $\eta_i = \iota \circ h_i$ onto $\tilde{S}$, and we can pull back λ and μ to measured laminations on S by the homeomorphism η_i^{-1}. Let λ_i, μ_i be measured laminations on S obtained as such pull-backs of λ and μ. As the projective lamination space is compact, there exist sequences of positive numbers $\{a_i\}, \{b_i\}$ such that $\{a_i\lambda_i\}$ and $\{b_i\mu_i\}$ converge to measured laminations λ_∞ and μ_∞ respectively.

We shall complete the proof of Lemma 3.4 by proving the following two claims.

Claim 1. Let $\tilde{f}_\infty : S \to \tilde{M}_\infty$ be a lift of f_∞ such that $\tilde{f}_{\infty\#}\pi_1(S) = \pi_1(\tilde{M}_\infty)$. Then there are no pleated surfaces homotopic to $\tilde{f}_\infty$ which realize λ_∞ or μ_∞. Furthermore, both λ_∞ and μ_∞ are maximal (i.e. not contained in other measured laminations as proper sublaminations) and connected.

Proof. We shall prove the first part of this claim by contradiction. Suppose first that there is a pleated surface homotopic to $\tilde{f}_\infty$ which realizes λ_∞. Then by composing the covering projection, we get a pleated surface into M_∞ homotopic to f_∞ which realizes λ_∞. Then by Lemmata 4.6, 6.9 in [24], we can see that for sufficiently large i, the measured lamination λ_i can be realized by a pleated surface homotopic to $\rho_i^{-1} \circ f_\infty$, which implies that λ_0 is realized by a pleated surface homotopic to $\iota : S \to M_\infty$. This contradicts the assumption that λ is an ending lamination.

Next suppose that there is a pleated surface homotopic to $\tilde{f}_\infty$ which realizes μ_∞. Then by composing the covering projection, we get a pleated surface $g_\infty : S \to M_\infty$ homotopic to f_∞ which realizes μ_∞. Again by Lemmata 4.6, 6.9 in [24], we can see that for any $\epsilon > 0$, if we take a sufficiently large i, a pleated surface homotopic to ι which realizes μ_0 must intersect the ϵ-neighbourhood of $\rho_i^{-1}g_\infty(S)$. Since ρ_i^{-1} maps x_∞ to x_i and x_i goes to $\tilde{e}$ as $i \to \infty$, the surface $\rho_i^{-1}g_\infty(S)$ also tends to the end $\tilde{e}$ as $i \to \infty$. Hence μ is an ending lamination of $\tilde{e}$. This contradicts the assumption that μ_0 is realized by the pleated surface g which is homotopic to ι.

Now we prove the latter part of the claim. As is shown in Thurston [28] (see also Ohshika [22]), if there is a non-realizable measured lamination on S (with respect to pleated surfaces homotopic to f_∞) which is either non-maximal or disconnected, then there must be a parabolic element represented by a closed curve on $f_\infty(S)$. This cannot happen because $\tilde{\Gamma}_\infty$ has no parabolic elements. Hence both λ_∞ and μ_∞ must be maximal and connected. □

Claim 2. $i(\lambda_\infty, \mu_\infty) \neq 0$, where i denotes the geometric intersection number on S.

Proof. Since both λ_∞ and μ_∞ are maximal and connected by Claim 1, if $i(\lambda_\infty, \mu_\infty) = 0$, their supports must coincide. (This is derived from the maximality of ending laminations proved in Thurston [28].)

Since $\eta_i = \iota \circ h_i$, by the definitions of λ_i and μ_i, we have $h_i^{-1}(\lambda_0) = \lambda_i$ and $h_i^{-1}(\mu_0) = \mu_i$. Since we assumed that $\{a_i\lambda_i\}$ and $\{b_i\mu_i\}$ converge to measured laminations with the same support, we can connect $[\mu_i]$ and $[\lambda_i]$ by an arc $\alpha_i : I = [0,1] \to \mathcal{PL}(S)$ $(\alpha_i(0) = [\mu_i], \alpha(1) = [\lambda_i])$ which converges to an arc $\alpha_\infty : I \to$

$\mathcal{PL}(S)$ uniformly such that $\alpha_\infty(t)$ has the same support as λ_∞ for any $t \in I$. Then $h_i \circ \alpha_i$ is an arc connecting $[\mu_0]$ and $[\lambda_0]$ in $\mathcal{PL}(S)$.

Recall that $\lambda = \iota(\lambda_0)$ is an ending lamination of the end $\tilde{e}$. Since $\tilde{M}$ has no cuspidal part and ending laminations are maximal and connected (see Thurston [28] for the case of hyperbolic 3-manifold, and see Canary [3] to learn how to generalize it to the case of negatively curved 3-manifold), no ending laminations of ends other than $\tilde{e}$ are homotopic into $\tilde{S}$. Now we should evoke the following lemma which was first proved by Bonahon as Proposition 5.1 in [2] in the case of hyperbolic 3-manifold, and then generalized by Canary to the case of negatively curved 3-manifold with pinched curvature (Proposition 2.2 in [3]). We have eliminated the mention of train tracks from the statement, as it is unnecessary for our purpose.

Lemma 3.5. *Let $\tilde{M}$ and S be as above with an homeomorphism $\iota : S \to \tilde{S} \subset \tilde{M}$. For any measured lamination ν on S, one (and only one) of the following two alternatives holds.*

(1) *For any $\epsilon > 0$, there exists a continuous map $f^\nu_\epsilon : S \to \tilde{M}$ homotopic to ι such that $\mathrm{length}(f^\nu_\epsilon(\nu)) < \epsilon$.*
(2) *For any $\epsilon > 0$ and $t < 1$, there exists a continuous map $f^\nu_{(\epsilon,t)} : S \to \tilde{M}$ homotopic to ι such that if γ is a simple closed curve on S whose projective class $[\gamma]$ is close to $[\nu]$ in $\mathcal{PL}(S)$, then the closed geodesic γ^* homotopic to $\iota(\gamma)$ has a part of length at least $t\mathrm{length}(f(\gamma))$ which is contained in the ϵ-neighbourhood of $f(\gamma)$.*

The first alternative holds if and only if $\iota(\nu)$ is an ending lamination. Moreover, if the second alternative holds and the image of $f^\nu_{(\epsilon,t)}$ does not intersect the part of $\tilde{M}$ where the curvature varies, then the map $f^\nu_{(\epsilon,t)}$ can be taken to converge to a pleated surface realizing ν as $\epsilon \to 0, t \to 1$.

Let ν be a measured lamination on S whose support is different from an ending lamination of λ_0. Then by the uniqueness of ending lamination, $\iota(\nu)$ is not an ending lamination of the end $\tilde{e}$, and cannot be an ending lamination of other ends, as was shown just before the lemma. It follows that the alternative (2) of Lemma 3.5 holds for such a measured lamination. On the other hand, for a measured lamination λ' which has the same support as λ, the alternative (1) holds.

We shall prove that if λ' has the same support as λ_0, then for any small neighbourhood $\check{E}$ of $\tilde{e}$, there is a neighbourhood U of λ' such that if $\nu \in U$ and the support of ν is different from that of λ_0, then the image of $f^\nu_{(\epsilon,t)}$ is contained in $\check{E}$ for sufficiently small ϵ, and t sufficiently close to 1. In particular, there is a pleated surface homotopic to ι realizing ν which is contained in $\check{E}$. The former fact can be proved as follows by contradiction. Suppose that there is a sequence of measured laminations $\{\nu_i\}$ converging to λ', for which the alternative (2) in Lemma 3.5 holds and such that $f^{\nu_i}_{(\epsilon,t)}(S)$ intersects a compact set K. Since we can choose maps $f^{\nu_i}_{(\epsilon,t)}$ so that their diameters are bounded by a universal constant by the same argument as one for bounding the diameters of pleated surfaces without thin parts (see Canary-Epstein-Green [5] for the proof), we can assume that there is a compact set K containing $f^{\nu_i}_{(\epsilon,t)}(S)$ for all i if we take small ϵ and t close to 1. Then there is a simple closed curve γ_i on S whose projective class is near to that of ν_i such that the closed geodesic γ^*_i has a part contained in K with length at least $t\mathrm{length}(f^{\nu_i}_{(\epsilon,t)}(\gamma_i))$. Since we can assume that $f^{\nu_i}_{(\epsilon,t)}$ converges to an incompressible surface uniformly after taking a subsequence, $\mathrm{length}(f^{\nu_i}_{(\epsilon,t)}(\gamma_i))/\mathrm{length}_S(\gamma_i)$ cannot go to 0 as $i \to \infty$. These imply that $\mathrm{length}(\gamma_i^*)/\mathrm{length}(\gamma_i)$, which should be greater than $t\mathrm{length}(f^{\nu_i}_{(\epsilon,t)}(\gamma_i))/\mathrm{length}_S(\gamma_i)$, cannot go to 0. This contradicts the assumption that the alternative 1 of Lemma 3.5 holds for λ_i and the fact that the projective classes $[\gamma_i]$ converge to $[\lambda']$.

Recall that $h_i \circ \alpha_i$ connects $[\mu_0]$ and $[\lambda_0]$ and that μ_0 has support different from that of λ_0. Let s_i be a number in I which is minimal among $s \in I$ such that alternative 1 of Lemma 3.5 holds for $h_i\alpha_i(s)$. (Note that the alternatives 1,2 do not depend on transverse measures, hence are well-defined for projective laminations, and that it is possible that $s_i = 1$, but always $s_i > 0$ since μ_0 is realized by g.) Then for any neighbourhood $\check{E}$ of $\tilde{e}$, if $s < s_i$ is sufficiently close to s_i, then $h_i\alpha_i(s)$ is realized by a pleated surface whose image is contained in $\check{E}$. Note that pleated surfaces realizing $h_i\alpha_i(s)$ may not move continuously with respect to s, but there is a universal upper bound C for the diameter of gaps at points of discontinuity. (This is proved by the same argument as before by interpolating negatively curved surfaces. See Thurston [28].) Hence there is a neighbourhood $\dot{E}$ of $\tilde{e}$ such that for any point $z \in \dot{E}$ and for each i, there is a pleated surface homotopic to ι realizing $h_i\alpha_i(s)$ for some $s \in I$ within distance C from z.

Now by Claim 1, we know that $\tilde{M}_\infty$ has at least one geometrically tame end $\tilde{e}_\infty$. By the covering theorem due to Thurston [28] (see also Canary [3], Ohshika [23]), there is a neighbourhood $\tilde{E}_\infty$ of $\tilde{e}_\infty$ homeomorphic to $S \times \mathbf{R}$ such that $q|\tilde{E}_\infty$ is a finite-sheeted covering. In our situation, this finite-sheeted covering must be one-to-one, for the following reason. If $q|\tilde{E}_\infty$ is not one-to-one, there is an incompressible map from a closed surface $g_\infty : T \to M_\infty$ such that $g_{\infty\#}\pi_1(T)$ is a finite-indexed proper subgroup of $f_{\infty\#}\pi_1(S)$. By composing the approximate isometry ρ_i^{-1}, this implies that there is an incompressible map $g_i : T \to \tilde{M}$ such that $g_{i\#}\pi_1(T)$ is a finite-indexed proper subgroup of $\pi_1(\tilde{S})$. This cannot happen, because $\tilde{S}$ is an embedded surface. Thus $q|\tilde{E}_\infty$ is a homeomorphism onto its image.

Let D be a universal upper bound for the diameters of pleated surfaces homotopic to f_∞ in M_∞. Fix a point $y \in q(\tilde{E}_\infty)$ such that the $(D + C)$-ball centred at y is contained in $\tilde{E}_\infty$. As was shown above, for sufficiently large i, there is a pleated surface $k_i : S \to \tilde{M}$ homotopic to ι, which realizes $h_i\alpha_i(t_i)$ for some $t_i \in I$, within distance C from $\rho_i^{-1}(y)$. (Note that since x_i tends to $\tilde{e}$, so does $\rho_i^{-1}(y)$.) Then k_i converges geometrically to a pleated surface $k_\infty : S \to M_\infty$ within distance C from y, as before. There is a homeomorphism $h : S \to S$ such that the pleated surface $k_\infty \circ h$ is homotopic to f_∞, because $k_\infty(S) \subset q(\tilde{E}_\infty)$ and any incompressible map from S to $q(\tilde{E}_\infty) \cong S \times \mathbf{R}$ is homotopic to a homeomorphism onto $S \times \{0\}$. We shall prove that h can be taken to be the identity. Since $k_\infty \circ h$ is homotopic to f_∞, by pulling back a homotopy by the approximate isometry ρ_i^{-1}, we can see that $k_i \circ h$ is homotopic to f_i for sufficiently large i. As k_i is homotopic to f_i (because both of them are homotopic to ι), and f_i is incompressible in $\tilde{M}$, the homeomorphism h must be homotopic to the identity. Thus we can assume h to be the identity. For each large i, there is a pleated surface $k_i' : S \to M_\infty$, approximated by k_i via ρ_i, which converges to k_∞ uniformly as $i \to \infty$ and is such that if ν_i is a measured lamination realized by k_i', then k_i realizes a measured lamination whose image is homotopic to $\rho_i^{-1}k_i'(\nu_i)$. (This pleated surface can be constructed by perturbing $\rho_i \circ k_i$ for sufficiently large i.) Then k_i' is homotopic to f_∞ for sufficiently large i. Hence $h_i(\nu_i)$ is realized by k_i if ν_i is realized by k_i'.

Therefore k_i' realizes $h_i^{-1}h_i\alpha_i(t_i) = \alpha_i(t_i)$. The pleated surface k_∞ realizes the limit of (a subsequence of) $\{\alpha_i(t_i)\}$, which is contained in $\alpha_\infty(I)$. This must have the same support as λ_∞ by assumption. This contradicts the fact that λ_∞ is an ending lamination. $\square$

Conclusion of the Proof of Lemma 3.4. We have already proved that $\tilde{\Gamma}_\infty$ has no parabolic elements, hence $\tilde{M}_\infty$ has two ends. By Claims 1, 2, and the uniqueness of the support of ending laminations, we can see that $\tilde{M}_\infty$ has two geometrically infinite tame ends. This means by definition that $\tilde{\Gamma}_\infty$ is a totally doubly degenerate group without parabolic elements. $\square$

Conclusion of the Proof of Lemma 3.3. It suffices to prove that $\tilde{M}_\infty = M_\infty$. As we have proved that $\tilde{\Gamma}_\infty$ is a totally doubly degenerate group, both ends of $\tilde{M}_\infty$ are geometrically infinite tame, and by Thurston's covering theorem either q is a finite-sheeted covering or M_∞ is compact. The latter case cannot occur since $\tilde{M}$ is not compact. By the same argument as in the proof of Claim 2, we can see that if q is a finite-sheeted covering, it must be a homeomorphism, using the facts that $\tilde{S}$ is embedded and that $\rho_i^{-1}f_\infty$ is homotopic to a homeomorphism onto $\tilde{S}$. Thus we have proved $\pi_1(M_\infty) = f_\# \pi_1(S)$, and the proof is completed. $\square$

The following corollary has been proved in the proof of Lemma 3.3

Corollary 3.6. *There exist a universal constant K and a constant Δ_0 depending only on $\tilde{M}$ such that if x is a point in $\tilde{M}$ apart from $\mathcal{N}(\tilde{\gamma})$ at distance greater than Δ_0 and contained in the component of the complement of $\tilde{C}$ containing an end which is projected to a geometrically infinite end of M, then x is contained in a neighbourhood of an end which is homeomorphic to $F \times \mathbf{R}$ for a closed surface F, and there exists a pleated surface homotopic to $F \times \{t\}$ within distance K from x.*

Next we shall prove that for a harmonic map into $\tilde{M}$ with high energy, there is a pleated surface near it which realizes a measured lamination corresponding to the horizontal foliation of the Hopf differential of the harmonic map. This is a generalization of Theorem 4.2 in Minsky [18].

Lemma 3.7. *There exist constants $\mathcal{E}_0, \eta, \Delta_1$ depending on $\chi(S)$ and $\tilde{M}$ as follows. Let $f : S \to \tilde{M}$ be a harmonic map whose energy is greater than $\mathcal{E}_0$ and whose image is apart from $\tilde{\gamma}$ at distance greater than Δ_1. Let μ be the measured lamination equivalent to the horizontal foliation associated to the Hopf differential of f, which is called the maximal stretch lamination of f. Then there is a pleated surface realizing μ whose image is within distance η from $f(S)$.*

Proof. The proof of this lemma is obtained by following the argument of the proof of Theorem 4.2 in Minsky [18]. We shall explain why the argument in [18] in the case when the target manifold is hyperbolic can be applied even to our situation where the sectional curvature varies near $\tilde{\gamma}$, if we assume that $f(S)$ is sufficiently far from $\tilde{\gamma}$.

Roughly speaking, Theorem 4.2 of [18] was proved there as follows. Let Φ_h be the horizontal measured foliation of the Hopf differential Φ for a given harmonic map. Let $|\phi|$ denote the Euclidean metric induced from the Hopf differential Φ, which is a quadratic differential. First, in Theorem 3.5 in [18], Minsky proved that at a point on S sufficiently far from the zeros of Φ with respect to the metric $|\Phi|$, the geodesic curvature of the image of leaves of Φ_h is bounded by $c\epsilon$ for some fixed point x on points where the values of the function $\mathcal{G}$ is bounded by ϵ. Here $\mathcal{G}$ is a function such that the pull-back of the metric of $\tilde{M}$ is expressed as $2(\cosh\mathcal{G}+1)dx^2+2(\cosh\mathcal{G}-1)dy^2$ with respect to the Euclidean metric $|\Phi|$. This gives rise to an approximation of $f(\Phi_h)$ by a train track with a small curvature which is located near $f(\Phi_h)$. By straightening the train track, Minsky obtained a realization of the measured lamination equivalent to Φ_h by a pleated surface. At this stage, Thurston's uniform injectivity theorem was used. The statement of Thurston's uniform injectivity theorem is as follows. (We describe it only in the special case when a map induces an isomorphism of the fundamental groups. This is sufficient for our case.) *Fix $\epsilon_0 > 0$ and a closed surface S. Let $f : S \to M$ be a pleated surface to a hyperbolic 3-manifold M realizing a geodesic lamination λ on S such that $f_\#$ induces an isomorphism of the fundamental groups. Let $g : \lambda \to UT(M)$ be the lift of $f|\lambda$ to the unit tangent bundle $UT(M)$ of M. Then for any ϵ, there exists $\delta > 0$, which depends only on ϵ and the topological type of S, such that for any two points $x, y \in \lambda$ which are in the ϵ_0-thick part of M such that $d(x, y) \geq \epsilon$, we have $d(g(x), g(y)) > \delta$, where the letter d denotes the metric on the unit tangent bundle induced from the hyperbolic metric on M.*

All the processes in the proof of Theorem 4.2 in [19] take place within a distance from $f(S)$ bounded by a constant depending only on $\chi(S)$. The uniform injectivity theorem also holds for a pleated surface to $\tilde{M}$ far from $\tilde{\gamma}$ which is homotopic to ι if we once fix $\tilde{M}$ and ι by virtue of Lemma 3.3, as will be proved below. Suppose that the uniform injectivity does not hold for such pleated surfaces. Then there exist a sequence of pleated surfaces $f_i : S \to \tilde{M}$ homotopic to ι realizing a geodesic lamination λ_i which tends to $\tilde{e}$ as $i \to \infty$, and points $x_i, y_i \in \lambda_i$ such that $d(x_i, y_i) > \epsilon > 0$ but $d(g_i(x_i), g_i(y_i)) \to 0$, where g_i is the lift of $f_i|\lambda_i$ to the unit tangent bundle of $\tilde{M}$. Then after taking a subsequence, the sequence $\{f_i\}$ converges to a pleated surface $f_\infty : S \to M_\infty$, the geodesic lamination λ_i converges geometrically to a geodesic lamination λ_∞, and there are points $x_\infty, y_\infty \in \lambda_\infty$ to which x_i, y_i converge respectively as $i \to \infty$. For the lift g_∞ of $f_\infty|\lambda_\infty$ to the unit tangent bundle of M_∞, we have $g_\infty(x_\infty) = g_\infty(y_\infty)$. This contradicts Thurston's uniform injectivity theorem, because we have proved that M_∞ is a hyperbolic 3-manifold whose fundamental group is isomorphic to $\pi_1(S)$, which implies that f_∞ induces an isomorphism of the fundamental groups. Thus we can use the uniform injectivity theorem in the last step of the proof, as in the proof of Theorem 4.2 in [19]. Hence the argument as described above can be applied to our case if we choose sufficiently large Δ_0. □

Entirely by the same argument as in Minsky [18], using the above lemma, we can prove the following inequality.

Lemma 3.8. *There exist constants Δ_2 and d depending only on $\chi(S)$ and $\tilde{M}$ as follows. Let $f : (S, \sigma) \to \tilde{M}$ be a harmonic map whose image is apart from $\mathcal{N}(\tilde{\gamma})$ at a distance greater than Δ_2. Let μ be the measured lamination equivalent to the horizontal measured foliation of the Hopf differential of f. Then*

$$\frac{\mathrm{length}^2_{\tilde{M}}(f(\lambda)^*)}{2E_\sigma(\lambda)} \leq \mathcal{E}(f) \leq \frac{\mathrm{length}^2_{\tilde{M}}(f(\mu)^*)}{2E_\sigma(\mu)} + d$$

for any measured lamination λ on S, where E denotes the extremal length and $f(\lambda)^*$ denotes the realization of λ by a pleated surface homotopic to f.

For $\lambda, \mu \in \mathcal{ML}(S)$, which can be realized by pleated surfaces homotopic to $\iota : S \to \tilde{S}$, we define

$$I_{\tilde{M}}(\lambda, \mu) = \frac{i(\lambda, \mu)}{\text{length}_{\tilde{M}}(\lambda^*)\text{length}_{\tilde{M}}(\mu^*)},$$

where $\text{length}_{\tilde{M}}(\lambda^*)$ denote the length of the realization of λ by a pleated surface homotopic to the inclusion. Similarly, for a hyperbolic structure σ on S, we define $I_\sigma(\lambda, \mu)$ to be the quotient of $i(\lambda, \mu)$ by $\text{length}_\sigma(\lambda)\text{length}_\sigma(\mu)$. The following is an adaptation of Lemma 5.2 in our situation.

Lemma 3.9. *There exist positive constants D_1 and C_1 as follows. For two measured laminations λ, μ which can be realized by pleated surfaces f_1, f_2 homotopic to $\iota : S \to \tilde{M}$, if $d(f_1(\lambda), f_2(\mu)) \geq D_1$ then $I_{\tilde{M}}(\lambda, \mu) \geq C_1$.*

Proof. Our proof is quite similar to that of Lemma 5.2 in Minsky [19]. We shall prove this lemma by reductio ad absurdum. Suppose that there exist two sequences $\{f_1^i\}$ and $\{f_2^i\}$ of pleated surfaces realizing λ_i and μ_i such that $d(f_1^i(\lambda_i), f_2^i(\mu_i)) \to \infty$ and $I_{\tilde{M}}(\lambda_i, \mu_i) \to 0$. Then by taking subsequences, we can assume that the distance from $\mathcal{N}(\tilde{\gamma})$ to either $f_1^i(S)$ or $f_2^i(S)$, say $f_1^i(S)$, goes to infinity as $i \to \infty$. Since $\tilde{M}$ is the interior of the acylindrical 3-manifold $\overline{\tilde{M}}$, the pleated surfaces f_1^i cannot go out from $\tilde{C}$ to a component of the complement other than one facing S. Hence f_1^i tends to the end $\tilde{e}$ facing S as $i \to \infty$. Put a base frame v_i on $f_1^i(S)$ and consider the geometric limit of $\{(\tilde{M}, v_i)\}$. Then by Lemma 3.3, $\{(\tilde{M}, v_i)\}$ converges to a hyperbolic 3-manifold (M_∞, v_∞) with fundamental group isomorphic to $\pi_1(S)$.

Since f_1^i is incompressible and the injectivity radius at any point of $\tilde{M}$ is bounded below by ϵ_0, the hyperbolic structure σ_1^i induced on S by f_1^i is bounded in the moduli space as $i \to \infty$. Hence $f_1^i : S \to \tilde{M}$ converges geometrically to a pleated surface $f_1^\infty : (S, \sigma_1^\infty) \to M_\infty$ as is shown in Thurston [31], and there are approximate isometries $\overline{\rho}_i : (S, \sigma_1^i) \to (S, \sigma_1^\infty)$ and $\rho_i : B_{r_i}(\tilde{M}, x_i) \to B_{r_i}(M_\infty, x_\infty)$, where $r_i \to \infty$, such that $\rho_i \circ f_1^i = f_1^\infty \circ \overline{\rho}_i$.

Recall that f_1^i realizes λ_i. Let $[\lambda_\infty]$ be a projective lamination which is a limit of $\{[\overline{\rho}_i(\lambda)]\}$ after taking a subsequence. Then the pleated surface f_1^∞ realizes λ_∞.

By assumption, we have $I_{\sigma_1^i}(\lambda_i, \mu_i) \geq I_{\tilde{M}}(\lambda_i, \mu_i) \to 0$. Since I is invariant under the scalar multiplication, we can assume that $\text{length}_{\sigma_1^i}(\lambda_i) = \text{length}_{\sigma_1^i}(\mu_i) = 1$. Then $\{\overline{\rho}_i(\lambda_i)\}$ and $\{\overline{\rho}_i(\mu_i)\}$ converges to measured laminations λ_∞, μ_∞ whose lengths with respect to σ_1^∞ are 1. As $i(\overline{\rho}_i(\lambda_i), \overline{\rho}_i(\mu_i)) = i(\lambda_1, \mu_1) = I_{\sigma_1^i}(\lambda_1, \mu_i) \to 0$, we have $i(\lambda_\infty, \mu_\infty) = 0$. This implies that there exists a pleated surface into M_∞ homotopic to f_1^∞ realizing $\lambda_\infty \cup \mu_\infty$, because ending laminations are maximal and connected, and any measured lamination which is not an ending lamination can be realized by a pleated surface homotopic to f_1^∞. It follows from the argument in the proof of Proposition 4.8 in Ohshika [24] that μ_i must be realized by a pleated surface whose distance from f_1^i is bounded by an upper bound of the diameters of pleated surfaces in $\tilde{M}$. This contradicts our assumption that $d(f_1^i(\lambda_i), f_2^i(\mu_i)) \to \infty$, because the realized image of μ_i does not depend on a pleated surface realizing it. $\square$

Lemma 3.10. *There exists a constant D_2 as follows. For any $a > 0$, there exists $b > 0$ such that if $f_1 : (S, \sigma_1) \to \tilde{M}$ and $f_2 : (S, \sigma_2) \to \tilde{M}$ are two pleated surfaces homotopic to $\iota : S \to \tilde{M}$ which are apart from $\mathcal{N}(\tilde{\gamma})$ at distance greater than D_2 and such that $d_{\tilde{M}}(f_1(S), f_2(S)) \leq a$, then $d_T(\sigma_1, \sigma_2) \leq b$.*

Proof. This lemma is an analogue of Corollary 4.6 in Minsky [19]. It is proved there that for any $a > 0$ there exists $b > 0$ as follows. In any hyperbolic 3-manifold N whose fundamental group is isomorphic to $\pi_1(S)$ and in which the injectivity radius is bounded below by ϵ_0 at any point, if $f : (S, \rho) \to N$ and $g : (S, \tau) \to N$ are two homotopic pleated surfaces such that $d_N(f(S), g(S)) \leq a$, then $d_T(\rho, \tau) \leq b$. Our method of proving this lemma is quite similar to that of Corollary 4.6.

First we shall prove the following lemma, which is an analogue of Lemma 4.5 in Minsky [19].

Lemma 3.11. *There exists a neighbourhood $\tilde{E}$ of the end $\tilde{e}$ as follows. For given positive constants b_1, b_2, there exists a constant A such that if $f : (S, \sigma) \to \tilde{E}$ is a pleated surface homotopic to $\iota : S \to \tilde{M}$, α is a closed geodesic on (S, σ), and $\beta \subset \tilde{E}$ is the shortest closed curve homotopic to $f(\alpha)$ within distance b_1 from $f(S)$, then the condition that* $\mathrm{length}_{\tilde{M}}(\beta) \leq b_2$ *implies that* $\mathrm{length}_\sigma(\alpha) \leq A$.

Proof. Fix a neighbourhood $\tilde{E}$ of $\tilde{e}$ whose closure is homeomorphic to $S \times [0, 1)$ and disjoint from $\mathcal{N}(\gamma)$.

Suppose that this lemma is not true. Then there exist pleated surfaces $f^i : (S, \sigma^i) \to \tilde{M}$, closed geodesics α^i on (S, σ^i), and the shortest closed curves β^i homotopic to $f^i(\alpha^i)$ within distance b_1 from $f^i(\alpha^i)$ such that $\mathrm{length}_{\tilde{M}}(\beta^i) \leq b_2$ but $\mathrm{length}_{\sigma^i}(\alpha^i) \to \infty$. Put a base frame v_i on $f^i(S)$ for each i. There are two possibilities. The one is that v_i remains in a compact set after taking a subsequence as $i \to \infty$. The other is that v_i tends to the end $\tilde{e}$ as $i \to \infty$. Consider the former case first. Then the geometric limit of $\{(\tilde{M}, v_i)\}$, after taking a subsequence, is $(\tilde{M}, v_\infty)$ for some base frame v_∞ on a point in $\tilde{M}$. Since the diameters of $f^i(S)$ are uniformly bounded above, the distance from the basepoint x_i, on which v_i lies, to the closed curve β^i is bounded above as $i \to \infty$. It follows that β^i converges to a closed curve β^∞, as the length of β^i is bounded above by b_2. Because β^i is contained in $\tilde{E}$, the limit β^∞ is contained in the closure of $\tilde{E}$. On the other hand, the pleated surfaces $f^i : (S, \sigma^i) \to \tilde{M}$ converge to a pleated surface $f^\infty : (S, \sigma^\infty) \to \tilde{M}$, because the diameters of these pleated surfaces are uniformly bounded. As the pleated surfaces f^i are incompressible and contained in the closure of $\tilde{E}$, so is the limit f^∞. This implies that f^∞ is homotopic to a surface of the form $S \times \{t\}$ when we regard the closure of $\tilde{E}$ as $S \times [0, 1)$. Thus there exists a closed geodesic α' on (S, σ^∞) such that $f^\infty(\alpha')$ is homotopic to β^∞. As $\{(\tilde{M}, f^i, v_i)\}$ converges geometrically to $(\tilde{M}, f^\infty, v_\infty)$, there are approximate isometries $\rho_i : B_{r_i}(\tilde{M}, v_i) \to B_{r_i}(\tilde{M}, v_\infty)$ and $\overline{\rho}_i : (S, \sigma^i) \to (S, \sigma^\infty)$ such that $\rho_i \circ f^i = f^\infty \circ \overline{\rho}_i$. By pulling back the homotopy between $f^\infty(\alpha')$ and β^∞ by ρ_i, we get a homotopy between $f^i(\rho_\infty^{-1}(\alpha'))$ and β^i. It follows that $\rho_\infty^{-1}(\alpha')$ is homotopic to α^i on S. This contradicts the assumption that $\mathrm{length}_{\sigma^i}(\alpha^i) \to \infty$.

Next assume that $\{v_i\}$ tends to the end $\tilde{e}$. By Lemma 3.3, $\{(\tilde{M}, v_i)\}$ converges to a hyperbolic 3-manifold (N, v_∞) whose fundamental group is isomorphic to $\pi_1(S)$. We can see that the pleated surfaces $f^i : (S, \sigma^i) \to \tilde{M}$ and the closed curves β^i converge to a pleated surface $f^\infty : (S, \sigma^\infty) \to N$ and a closed curve $\beta^\infty \subset N$ within distance b_1 from $f^\infty(S)$ whose length is at most b_2, by the same argument as in the first case. Since f^i is incompressible, so is f^∞. Hence $f^\infty_\# : \pi_1(S) \to \pi_1(N)$ is an isomorphism. Thus there is a closed geodesic α^∞ on (S, σ^∞) such that $f^\infty(\alpha^\infty)$ is homotopic to β^∞. Then, as before, using the approximate isometries, we can prove that there is a closed curve $\overline{\rho}_i^{-1}(\alpha^\infty)$ converging to α^∞ geometrically, which is mapped to a closed curve homotopic to β^i by f^i. This contradicts the assumption that $\text{length}_{\sigma^i}(\alpha^i) \to \infty$. $\square$

Conclusion of the Proof of Lemma 3.10. Having proved the above lemma, we can prove Lemma 3.10 in entirely the same way as the proof of Corollary 4.6 in Minsky [19]. We shall briefly review the proof. Let $X = \{\xi_1, \ldots, \xi_n\}$ be a binding collection of simple closed curves on S, i.e., simple closed curves such that every component of the complement of the union of the geodesic representatives of them is a disc. Let ξ_j^1 and ξ_j^2 be closed geodesics homotopic to ξ_j with respect to σ_1 and σ_2 respectively. For given σ_1, we can choose a binding collection X so that $\sum_j \text{length}_{\sigma_1}(\xi_j^1)$ is bounded above by a constant A depending on δ_0 but independent of f_1 and σ_1. Since the diameter $f_2(S)$ is universally bounded and $d(f_1(S), f_2(S)) \leq a$, the distance from $f_2(\xi_j^2)$ to $f_1(S)$ is universally bounded for each j. It follows by Lemma 3.11 that the length of $f_2(\xi_j^2)$ is bounded above independently of f_1, f_2, which implies that $\sum_j \text{length}_{\sigma_2}(\xi_j^2)$ is bounded above. By Lemma 4.7 in [19] this implies that σ_2 is at a distance from σ_1 bounded above by a constant depending only on a. $\square$

Proposition 3.12. *For any $D > 0$, there is a neighbourhood $\tilde{E}$ of the end $\tilde{e}$ of $\tilde{M}$ facing the boundary component $\tilde{S}$ of the core $\tilde{C}$ as follows. Let $f_j : (S, \sigma_j) \to \tilde{E}$ $(j = 1, 2)$ be pleated surfaces homotopic to $\iota : S \to \tilde{M}$. Let $\overline{\sigma_1\sigma_2}$ be the Teichmüller geodesic segment connecting σ_1 and σ_2. Then for any $\tau \in \overline{\sigma_1\sigma_2}$, the harmonic map $f_\tau : (S, \tau) \to \tilde{M}$ homotopic to ι is apart from $\mathcal{N}(\tilde{\gamma})$ at distance greater than D.*

We shall prove Proposition 3.12 after Lemma 3.15.

For any hyperbolic structure σ on S there exists a unique harmonic map homotopic to ι from (S, σ) to $\tilde{M}$, as is shown in Minsky [18]. We shall denote this harmonic map by f_σ from now on. Following Minsky [19], for $\sigma \in \mathcal{T}(S)$ we define a subset $\Pi(\sigma)$ in $\mathcal{T}(S)$ in the following way.

$$\begin{aligned}\Pi(\sigma) = \{\tau \in \mathcal{T}(S) | \text{ there exists a pleated surface from } (S, \tau) \text{ to } \tilde{M} \\ \text{which intersects the } K\text{-neighbourhood of } f_\sigma(S)\},\end{aligned}$$

where K is the constant given in Corollary 3.6.

This set $\Pi(\sigma)$ is non-empty if $f_\sigma(S)$ is apart from $\mathcal{N}(\tilde{\gamma})$ at distance greater than Δ_0 by Corollary 3.6. We also define a subset $\mathcal{L}_a$ of $\mathcal{T}(S)$ to be $\{\sigma | d_T(\sigma, \Pi(\sigma)) \leq a\}$. If $f_\sigma(S)$ is apart from $\mathcal{N}(\tilde{\gamma})$ at distance greater than $\Delta = \max\{\Delta_0, \Delta_1, \Delta_2\}$, then, using Lemmata 3.7, 3.8, 3.10, 3.11, the proof of Proposition 6.2 in [19] works and we have

$$d_T(\sigma, \Pi(\sigma)) - c \leq \frac{1}{2} \log \mathcal{E}(\sigma) \leq d_T(\sigma, \Pi(\sigma)) + c, \tag{1}$$

where c is a constant depending only on $\chi(S)$ and $\tilde{M}$.

Using this inequality, we can prove the following lemma, which is a generalization of Lemma 6.3 in [19]. The proof is the same as that of Lemma 6.3 in [19].

Lemma 3.13. *There exist constants D_3 and η_2, depending only on $\chi(S)$ and $\tilde{M}$, as follows. For any pleated surface $f : (S, \sigma) \to \tilde{M}$ which is apart from $\mathcal{N}(\tilde{\gamma})$ at distance greater than D_3, the distance between $f(S)$ and the image of the harmonic map $f_\sigma(S)$ homotopic to f is smaller than η_2.*

Moreover, there exists a constant B_0 such that, for any $\sigma \in \mathcal{T}(S)$, if $f_\sigma(S)$ is apart from $\mathcal{N}(\tilde{\gamma})$ at distance greater than D_3, then $\mathrm{diam}(\Pi(\sigma)) \leq B_0$, which follows from Lemma 3.1 in Minsky [19] and Lemma 3.10.

Therefore we can apply the same argument as the proof of Proposition 6.5 in [19], and we can see that there are constants a_0, a_1, a_2 such that if $f_\sigma(S)$ and $f_\tau(S)$ are apart from $\mathcal{N}(\tilde{\gamma})$ by more than the distance $D_3 + \Delta$, then

$$\Pi(\sigma) \subset \mathcal{L}_{a_0}, \tag{2}$$

$$d_T(\sigma, \Pi(\sigma)) \leq d_T(\sigma, \tau) + d_T(\tau, \Pi(\tau)) + a_1, \tag{3}$$

$$d_T(\sigma, \Pi(\sigma)) - a_2 \leq d_T(\sigma, \mathcal{L}_{a_0}) \leq d_T(\sigma, \Pi(\sigma)). \tag{4}$$

Now we can prove the following lemma, analogously to Theorem 7.1 in [19].

Lemma 3.14. *Assume that $D \geq D_1 + D_2 + D_3 + \Delta + 1$. Then there exist constants B_1 and B_2 as follows. Let σ, τ be hyperbolic structures such that both $f_\sigma(S)$ and $f_\tau(S)$ are apart from $\mathcal{N}(\tilde{\gamma})$ at a distance greater than D. Then, if $d_T(\sigma, \mathcal{L}_{a_0}) > B_1 + d_T(\sigma, \tau)$, then* $\mathrm{diam}(\Pi(\sigma) \cup \Pi(\tau)) < B_2$.

Proof. The proof is entirely the same as the proof of Theorem 7.1 in [19], using the facts that both $f_\sigma(S)$ and $f_\tau(S)$ are far from $\mathcal{N}(\tilde{\gamma})$ and that the arguments before this lemma can be applied to them. Let us explain points of the proof briefly. Suppose that there are not constants B_1, B_2 such that if $d_T(\sigma, \mathcal{L}_{a_0}) > B_1 + d_T(\sigma, \tau)$ then $\mathrm{diam}(\Pi(\sigma) \cup \Pi(\tau)) < B_2$. Then for any B_1, B_2, we can take points σ, τ so that $\mathrm{diam}(\Pi(\sigma), \Pi(\tau)) \geq B_2$ while $d_T(\sigma, \mathcal{L}_{a_0}) > B_1 + d_T(\sigma, \tau)$. By Lemma 3.10, this implies that $d_{\tilde{M}}(f_\sigma(S), f_\tau(S))$ can be made arbitrarily large. Let μ_σ, μ_τ be the maximal stretch laminations for f_σ, f_τ respectively (that is, the measured laminations corresponding to the horizontal measured foliations of Hopf differentials for f_σ, f_τ). By the inequalities (1), (4), we can choose B_1 so that $\mathcal{E}(\sigma)$ and $\mathcal{E}(\tau)$ are sufficiently large. Let g_σ, g_τ be pleated surfaces homotopic to ι realizing μ_σ, μ_τ, which are proved to exist in Lemma 3.7. Then Lemma 3.7 implies that $d_{\tilde{M}}(g_\sigma(S), g_\tau(S))$ is also large, and from Lemma 3.9, it follows that so is $I_{\tilde{M}}(\mu_\sigma, \mu_\tau)$. Thus we can apply Lemma 3.8 to obtain $\mathcal{E}(\sigma) \leq 2\mathcal{E}(\sigma) - d \leq \mathrm{length}^2(g_\sigma(\mu_\sigma))/E_\sigma(\mu_\sigma)$ if $\mathcal{E}(\sigma)$ is sufficiently large, and a similar inequality for τ. On the other hand, by Lemma 5.1 in Minsky [19], we have $i(\mu_\sigma, \mu_\tau))^2 \leq E_\sigma(\mu_\sigma) E_\sigma(\mu_\tau)$. Moreover, as is proved in Kerckhoff [13], we have $E_\tau(\mu_\tau) \leq e^{2d_T(\sigma,\tau)} E_\sigma(\mu_\tau)$. These imply

$$\mathcal{E}(\sigma)\mathcal{E}(\tau) \leq \frac{\mathrm{length}^2(g_\sigma(\mu_\sigma))\mathrm{length}^2(g_\tau(\mu_\tau))}{E_\sigma(\mu_\sigma)E_\tau(\mu_\tau)} \leq \frac{e^{2d_T(\sigma,\tau)}}{I^2_{\tilde{M}}(\mu_\sigma, \mu_\tau)}.$$

Similarly, we have

$$\mathcal{E}(\tau) \geq \frac{\mathrm{length}^2(g_\sigma(\mu_\sigma))}{2E_\tau(\mu_\sigma)} \geq \frac{\mathrm{length}^2(g_\sigma(\mu_\sigma))}{2e^{2d_T(\sigma,\tau)}E_\sigma(\mu_\sigma)} \geq \frac{\mathcal{E}(\sigma)}{2e^{2d_T(\sigma,\tau)}}.$$

Thus we obtain the inequality $\mathcal{E}(\sigma) \leq \sqrt{2} e^{2d_T(\sigma,\tau)}$. This, together with the inequalities (1), (4), implies that

$$d_T(\sigma, \mathcal{L}_{a_0}) \leq \frac{1}{2} \log \mathcal{E}(\sigma) + c \leq d_T(\sigma, \tau) + \log \sqrt{2} + c.$$

This cannot hold if we choose sufficiently large B_1. □

The following lemma is also proved in entirely the same way as Lemma 8.2 in [19], setting $k = 1$.

Lemma 3.15. *If $r \geq B_1+B_2$, if a geodesic segment $\overline{xy}$ lies outside the r-neighbourhood of $\mathcal{L}_{a_0}$, and if for each $\sigma \in \overline{xy}$, the image of the harmonic map $f_\sigma(S)$ is apart from $\mathcal{N}(\tilde{\gamma})$ at a distance greater than D in the statement of Proposition 3.12, then*

$$\operatorname{diam}(\Pi(x) \cup \Pi(y)) \leq \frac{1}{2} d_T(x,y) + B_2.$$

Proof of Proposition 3.12. We shall prove this proposition by reductio ad absurdum. First note that since the injectivity radii at all points of $\tilde{M}$ are bounded below by $\epsilon_0 > 0$ as in Lemma 3.2, there is an upper bound L for the diameters of pleated surfaces homotopic to ι and harmonic maps f_τ homotopic to ι for $\tau \in \mathcal{T}(S)$. The existence of a bound for the diameters of pleated surfaces follows immediately from the facts that ι is incompressible and that the induced metric on a pleated surface is hyperbolic. That for the diameters of harmonic maps follows from the fact that the area of the image of harmonic map in $\tilde{M}$ is bounded by a constant depending only on the topological type of S, as is proved in Theorem 3.2 in Minsky [18]. We can assume that $D > D_1 + D_2 + D_3 + 2L + \Delta + K + 1$, where K is the constant in Corollary 3.6, because the statement of the proposition means "for any large D". Fix such D and suppose that for any small neighbourhood $\tilde{E}$ of $\tilde{e}$, there exists a pleated surface $f_j : (S, \sigma_j) \to \tilde{E}, (j = 1, 2)$ such that for some τ on the Teichmüller geodesic segment $\overline{\sigma_1\sigma_2}$ connecting σ_1 and σ_2, the harmonic map $f_\tau : (S, \tau) \to \tilde{M}$ homotopic to ι intersects the D-neighbourhood of $\mathcal{N}(\tilde{\gamma})$. By Lemma 3.13, by taking $\tilde{E}$ far enough from $\mathcal{N}(\tilde{\gamma})$, we can assume that the image of neither f_{σ_1} nor f_{σ_2} intersects the D-neighbourhood of $\mathcal{N}(\tilde{\gamma})$.

Now let τ_1 and τ_2 be points on the geodesic segment $\overline{\sigma_1\sigma_2}$ nearest to σ_1 and σ_2 respectively among the points τ on $\overline{\sigma_1\sigma_2}$ such that the image of the harmonic map f_τ intersects the D-neighbourhood of $\mathcal{N}(\tilde{\gamma})$. By the definition of τ_1 and τ_2, we have $d(\mathcal{N}(\tilde{\gamma}), f_{\tau_j}(S)) = D$ for $j = 1, 2$.

Fix a pleated surface $f_0 : (S, \sigma_0) \to \tilde{M}$ whose image has a point at a distance from $\mathcal{N}(\tilde{\gamma})$ between $D - K$ and $D + K$, where K is a constant in Corollary 3.6. Such a pleated surface is proved to exist by Corollary 3.6. Let B be the diameter of the set of points within distance $D + K + L$ from $\mathcal{N}(\tilde{\gamma})$. Then if a pleated surface $g : (S, \tau) \to \tilde{M}$ homotopic to ι is within distance $D + K + L$ from $\mathcal{N}(\tilde{\gamma})$, then $d(f_0(S), g(S)) \leq B$. By Lemma 3.10, there exists a constant b such that for any such pleated surface $g : (S, \tau) \to \tilde{M}$ which is apart from $\mathcal{N}(\tilde{\gamma})$ at a distance greater than D_2, we have $d_T(\sigma_0, \tau) \leq b$. Recall that we chose τ_j so that $d(f_{\tau_j}(S), \mathcal{N}(\tilde{\gamma})) = D$. Hence for any pleated surface $g : (S, \tau) \to \tilde{M}$ intersecting the K-neighbourhood of $f_{\tau_j}(S)$, we have $D - K - 2L \leq d(g(S), \mathcal{N}(\tilde{\gamma})) \leq D + K + L$. The first half of the inequality implies that $d(g(S), \mathcal{N}(\tilde{\gamma})) > D_2$; hence, together with the second half of the inequality, we have $d_T(\sigma_0, \tau) \leq b$, and $d_T(\sigma, \tau_j) \leq d_T(\tau_j, \Pi(\tau_j)) + B_0 + b$. (Recall that B_0 is a bound for the diameter of $\Pi(\tau_j)$.) Recall that τ_1 and τ_2 lie on the Teichmüller geodesic segment whose endpoints are represented by hyperbolic structures of pleated surfaces. We need to prove the following two lemmata to continue our proof.

Lemma 3.16. *Let $\{g_i : (S, \nu_i) \to \tilde{M}\}$ be a sequence of pleated surfaces homotopic to ι tending to the end $\tilde{e}$. Then $\{\nu_i\}$ converges in the Thurston compactification of the Teichmüller space to a projective lamination which represents an ending lamination for $\tilde{e}$ after taking a subsequence.*

Remark 1. Although $\tilde{M}$ is not a hyperbolic 3-manifold, ending laminations are defined for ends of $\tilde{M}$ similarly to the case of hyperbolic 3-manifolds because each end has a neighbourhood where the curvature is constantly -1. Since $\tilde{M}$ is the interior of a boundary-irreducible 3-manifold, for each end, the support of ending laminations for the end is unique.

Proof. Let c_i be the shortest closed geodesic on S with respect to the hyperbolic structure ν_i. Let c_i^* be the closed geodesic in $\tilde{M}$ homotopic to $f_i(c_i)$. As the injectivity radii are bounded below by ϵ_0, and f_i is incompressible, the lengths of the geodesics c_i^* are bounded above by a universal constant depending only on $\chi(S)$. Again using the fact that the injectivity radii at all points of $\tilde{M}$ are bounded below by ϵ_0, we can see that the length of c_i^* is bounded below by $2\epsilon_0$; hence there is a constant c' such that $d(f_i(c_i), c_i^*) \leq c'$. Thus the sequence of closed geodesics $\{c_i^*\}$ tends to the end $\tilde{e}$. Note that there is no subsequence of $\{\nu_i\}$ which converges inside $\mathcal{T}(S)$, because f_i tends to an end. Let $[\nu]$ be a projective lamination to which $\{\nu_i\}$ converges in the Thurston compactification of the Teichmüller space. Let $[\mu]$ be a projective lamination to which the projective classes of c_i converge in the projective lamination space $\mathcal{PL}(S)$. Then by Lemma 3.4 in Ohshika [22], we have $i(\mu, \nu) = 0$. On the other hand, by definition, μ is an ending lamination for the end $\tilde{e}$. As ending laminations are maximal, it follows that the supports of μ and ν coincide. Thus ν is also an ending lamination, which completes the proof. $\square$

Lemma 3.17. *Let $\{s_i\}$ be a sequence of Teichmüller geodesic segments with endpoints $x(s_i)$ and $y(s_i)$ such that $\{x(s_i)\}$ and $\{y(s_i)\}$ converge respectively in the Thurston compactification of the Teichmüller space to maximal projective laminations $[\mu^1]$, $[\mu^2]$ which have the same support. Then for any compact set X in the Teichmüller space, $s_i \cap X$ is empty for sufficiently large i.*

Proof. Suppose that the lemma is false. Then there exists a subsequence of $\{s_i\}$ which converges uniformly on compact sets. We shall denote this convergent subsequence again by $\{s_i\}$. Let s_∞ be the Teichmüller geodesic line which is the limit of the sequence above. Fix a basepoint x_0 on s_∞; then s_∞ determines up to scalar multiples two measured foliations λ_∞, μ_∞ corresponding to the endpoints of s_∞ in the Teichmüller compactification of $\mathcal{T}(S)$ (i.e., λ_∞ and μ_∞ are horizontal and vertical foliations of the holomorphic quadratic differential defining the Teichmüller line s_∞). Similarly, let λ_i and μ_i be two measured foliations corresponding to the endpoints of the extension of s_i to a Teichmüller line after putting a basepoint x_i on s_i which converges to x_0. By definition, for any sequence $\{\sigma_k\}$ on s_∞ going to the end, we have either $E_{\sigma_k}(\lambda_\infty) \to 0$ or $E_{\sigma_k}(\mu_\infty) \to 0$. As we can apply the same argument to both cases, let us assume that $E_{\sigma_k}(\lambda_\infty) \to 0$. Since s_i converges to s_∞, and the extremal length is continuous with respect to the Teichmüller metric, we can find $\sigma'_j \in s_{i_j}$ such that $d_T(x_0, x_{i_j}) \leq 1/j$ and $E_{\sigma'_j}(\lambda_\infty) \to 0$ as $j \to \infty$. Either λ_{i_j} or μ_{i_j} (we can assume λ_{i_j}) is a stretching measured foliation from x_{i_j} to σ'_j. Then $E_{\sigma'_j}(\lambda_{i_j}) \to 0$ as $j \to \infty$, since $d_T(x_{i_j}, \sigma'_j) \to \infty$ as $j \to \infty$. Since $i(\lambda_\infty, \lambda_{i_j})^2 \leq E_{\sigma'_j}(\lambda_\infty) E_{\sigma'_j}(\lambda_{i_j})$ by Lemma 5.1 in Minsky [19], we can see that, after taking a subsequence and scalar multiplications, $\{\lambda_{i_j}\}$ converges to a measured foliation λ'_∞ such that $i(\lambda_\infty, \lambda'_\infty) = 0$.

Let y_{i_j} be an endpoint of s_{i_j} ($x(s_{i_j})$ or $y(s_{i_j})$) on the same side of x_{i_j} as σ'_j. Then $E_{y_{i_j}}(\lambda_{i_j}) \to 0$ as before, which implies that $\mathrm{length}_{y_{i_j}}(\lambda_{i_j}) \to 0$. By assumption $\{y_{i_j}\}$ converges to either $[\mu^1]$ or $[\mu^2]$, say $[\mu^1]$, in the Thurston compactification of the Teichmüller space. By Lemma 3.4 in [22], we have $i(\lambda_{i_j}, \mu^1) \to 0$, and by the maximality of μ^1, we have $|\lambda'_\infty| = |\mu^1|$, where $|\ |$ denotes the support. Then we can see that λ'_∞ is also maximal, and it follows that $|\lambda_\infty| = |\lambda'_\infty| = |\mu^1|$. The same argument implies that $|\mu_\infty| = |\mu^2|$ (after exchanging the names of μ^1 and μ^2 if necessary). As we assumed that supports of μ^1 and μ^2 coincide, this means that s_∞ is a Teichmüller geodesic line connecting two measured foliations with the same support in the Teichmüller compactification, which is impossible. Thus we have proved that $\{s_i\}$ cannot remain intersecting a compact set. $\square$

Conclusion of the Proof of Proposition 3.12. By Lemma 3.13, $d(f_j(S), f_{\sigma_j}(S))$ ($j = 1, 2$) is bounded by a constant depending on D but not on σ_j; hence we can bound $d_T(\sigma_j, \Pi(\sigma_j))$ and $d_T(\sigma_j, \mathcal{L}_{a_0})$ from above, using Lemma 3.10 and the inequality (4), by a constant r_0 independent of σ_j. Fix $r > r_0$ so that moreover $r \geq B_1 + B_2$, as in Lemma 3.15. Let m_1 and m_2 be points on the segments $\overline{\sigma_1\tau_1}$ and $\overline{\sigma_2\tau_2}$ such that $d_T(m_1, \mathcal{L}_{a_0}) = d_T(m_2, \mathcal{L}_{a_0}) \leq r$, and m_j is the nearest to τ_j among such points. Such points m_1, m_2 exist because $d_T(\sigma_j, \mathcal{L}_{a_0}) \leq r_0 < r$. We shall show that we can assume that $d_T(\tau_j, \mathcal{L}_{a_0}) > r$. This can be done by choosing $\tilde{E}$ sufficiently small, because $d_T(\tau_j, \mathcal{L}_{a_0}) \geq d_T(\tau_j, \Pi(\tau_j)) - a_2 \geq d_T(\sigma_0, \tau_j) - B_0 - b - a_2$ and the right hand side can be taken arbitrarily large choosing a small $\tilde{E}$ (as will be shown below). Consider the segment $\overline{\sigma_1\sigma_2}$. As $\tilde{E}$ becomes smaller and smaller, both σ_1 and σ_2, which are hyperbolic structures on pleated surfaces contained in $\tilde{E}$, tend to an ending lamination of $\tilde{e}$ in the Thurston compactification of the Teichmüller space, as is shown in Lemma 3.16. Then by Lemma 3.17 the segment $\overline{\sigma_1\sigma_2}$ can be taken arbitrarily far from the base point σ_0. This implies that $d_T(\sigma_0, \tau_j)$ can be taken arbitrarily large. Thus by the definition of m_1, m_2, we have $d_T(\overline{m_j\tau_j}, \mathcal{L}_{a_0}) \geq r$ for $j = 1, 2$.

Now since $d_T(m_j, \Pi(m_j)) \leq d_T(m_j, \mathcal{L}_{a_0}) + a_2$, we have

$$d_T(m_1, m_2) \leq \operatorname{diam}(\Pi(m_1) \cup \Pi(m_2)) + 2r + 2a_2.$$

On the other hand, we have

$$\begin{aligned}
\operatorname{diam}(\Pi(m_1) \cup \Pi(m_2)) &\leq \operatorname{diam}(\Pi(m_1) \cup \Pi(\tau_1)) + \operatorname{diam}(\Pi(\tau_1) \cup \Pi(\tau_2)) \\
&\quad + \operatorname{diam}(\Pi(\tau_2) \cup \Pi(m_2)) \\
&\leq \tfrac{1}{2}(d_T(m_1, \tau_1) + d_T(m_2, \tau_2)) + 2B_2 \\
&\quad + \operatorname{diam}(\Pi(\tau_1) \cup \Pi(\tau_2)) \\
&\quad \text{(by Lemma 3.15)} \\
&\leq \tfrac{1}{2} d_T(m_1, m_2) + 2B_2 + \operatorname{diam}(\Pi(\tau_1) \cup \Pi(\tau_2)).
\end{aligned}$$

We shall bound $\operatorname{diam}(\Pi(\tau_1) \cup \Pi(\tau_2))$ next. Since both $f_{\tau_1}(S)$ and $f_{\tau_2}(S)$ are at distance D from $\mathcal{N}(\tilde{\gamma})$, if $\nu, \nu' \in \Pi(\tau_1) \cup \Pi(\tau_2)$, then as before, there are $\nu_0 \in \Pi(\tau_1)$ and $\nu'_0 \in \Pi(\tau_2)$ such that $d_T(\nu_0, \nu'_0) \leq b$ by Lemma 3.10. Hence we have $d_T(\nu, \nu') \leq b + 2B_0$, which implies that $\operatorname{diam}(\Pi(\tau_1) \cup \Pi(\tau_2)) \leq b + 2B_0$. Thus $d_T(m_1, m_2) \leq \frac{1}{2} d_T(m_1, m_2) + 2B_2 + b + 2B_0 + 2r + 2a_2$, and hence

$$d_T(m_1, m_2) \leq 4B_2 + 2b + 4B_0 + 4r + 4a_2. \tag{5}$$

Next we shall show that there is a large lower bound for $d_T(m_1, m_2)$ by finding a lower bound for $d_T(m_j, \tau_j)$. Now,

$$\begin{aligned}
d_T(m_j, \tau_j) &\geq d_T(\tau_j, \Pi(\tau_j)) - d_T(m_j, \Pi(m_j)) \\
&\quad - \operatorname{diam}(\Pi(m_j) \cup \Pi(\tau_j)) \\
&\geq d_T(\tau_j, \Pi(\tau_j)) - r - a_2 - \tfrac{1}{2} d_T(m_j, \tau_j) - B_2
\end{aligned}$$

by Lemma 3.15. Hence

$$d_T(m_j, \tau_j) \geq \frac{2}{3}\{d_T(\tau_j, \Pi(\tau_j)) - r - a_2 - B_2\}.$$

Since $d_T(\tau_j, \Pi(\tau_j)) \geq d_T(\sigma_0, \tau_j) - B_0 - b$, as was shown before, the right hand side of the inequality above can be taken arbitrarily large by choosing small $\tilde{E}$. This contradicts the inequality (5), because $d_T(m_1, m_2) > d_T(m_j, \tau_j)$. □

Now we can prove our main theorem in this section, Theorem 3.1

Proof of Theorem 3.1. Let $p : \tilde{M} \to M$ be the branched covering map constructed before. For a neighbourhood $\tilde{E}$, which is sufficiently far from $\mathcal{N}(\tilde{\gamma})$, of the end $\tilde{e}$ of $\tilde{M}$, which is projected to the end e of M, the restriction $p|\tilde{E}$ is a proper homeomorphism to its image. Hence we have only to prove that we can choose a neighbourhood $\tilde{E}$ of $\tilde{e}$ satisfying the conditions (1)-(3) of the theorem.

In Proposition 3.12, we have proved that for any $D > 0$, if we choose a sufficiently small neighbourhood $\tilde{E}$ of $\tilde{e}$, then for any two pleated surfaces $f_1 : (S, \sigma_1) \to \tilde{E}$, $f_2 : (S, \sigma_2) \to \tilde{E}$ homotopic to ι, and any point $\sigma \in \overline{\sigma_1\sigma_2}$, the image the harmonic map f_σ is disjoint from the D-neighbourhood $\mathcal{N}(\tilde{\gamma})$. In particular, $f_\sigma(S)$ does not intersect $\mathcal{N}(\tilde{\gamma})$; hence it lies within the part of $\tilde{M}$ where the sectional curvature is constantly -1. This means that the whole argument of the proof of Theorem A in Minsky [19] can be applied to our case. Let us see briefly how the proof of Theorem A can be applied to complete our proof.

First, we can prove that there exists $r' > 0$ such that for any pleated surfaces $f_1 : (S, \sigma_1) \to \tilde{E}$ and $f_2 : (S, \sigma_2) \to \tilde{E}$, the geodesic segment $\overline{\sigma_1\sigma_2}$ is contained in $\mathcal{L}_{r'}$. This can be shown by the same argument as the proof of Theorem 8.1 in Minsky [19] using Lemma 3.15 and the inequality (4) in this paper.

Now fix a pleated surface $g_0 : (S, \tau_0) \to \tilde{E}$, and choose a sequence of pleated surfaces $g_i : (S, \tau_i) \to \tilde{E}$ tending to the end $\tilde{e}$. Let s_i be the Teichmüller geodesic segment joining τ_0 and τ_i. By the remark above, we have $s_i \subset \mathcal{L}_{r'}$. Since the segments s_i have the common endpoint τ_0, the sequence $\{s_i\}$ converges, with respect to the compact-open topology, to a Teichmüller geodesic ray $s_\infty : [0, \infty) \to \mathcal{T}(S)$ with $s_\infty(0) = \tau_0$ whose image is contained in $\mathcal{L}_{r'}$, and $f_{s_\infty(r)}(S)$ does not meet the D-neighbourhood of $\mathcal{N}(\tilde{\gamma})$ because we can apply Proposition 3.12 to converging segments s_i. Take a neighbourhood $\tilde{E}'$ of $\tilde{e}$ homeomorphic to $S \times \mathbf{R}$ which is contained in $\tilde{E}$ and such that $f_{\tau_0}(S) \cap \tilde{E}' = \emptyset$.

For this neighbourhood $\tilde{E}'$, let $h : (S, \eta) \to \tilde{E}'$ be a pleated surface. Note that by uniqueness of harmonic maps from Riemann surface into a negatively curved 3-manifold, the surface $f_{s_\infty(t)}(S)$ moves continuously with respect to t. Furthermore, since $s_\infty(t) \in \mathcal{L}_{r'}$, we have $d_T(s_\infty(t), \Pi(s_\infty(t))) \leq r'$. As $t \to \infty$, the hyperbolic structure $s_\infty(t)$ goes away to infinity in $\mathcal{T}(S)$, which implies that so does $\Pi(s_\infty(t))$. It follows that $f_{s_\infty(t)}(S)$ tends to an end in $\tilde{M}$, which must be $\tilde{e}$. Hence for some t, the surface $f_{s_\infty(t)}(S)$ must intersect $h(S)$; in other words, $h \in \Pi(s_\infty(t))$. This implies that $d_T(\eta, s_\infty) \leq r' + B_0$, since $\mathrm{diam}(\Pi(s_\infty(t)) \leq B_0$.

On the other hand, for any $\sigma \in s_\infty$, we have $d_T(\sigma, \Pi(\sigma)) \leq r'$ because $s_\infty \subset \mathcal{L}_{r'}$, and $\Pi(\sigma) \neq \emptyset$ because $d(f_\sigma(S), \mathcal{N}(\tilde{\gamma})) \geq D > \Delta_0$. These imply that there exists a pleated surface $h : (S, \tau) \to \tilde{M}$ such that $d_T(\sigma, \tau) \leq r'$. Hence by letting the projection of $\tilde{E}'$ to M be $\overline{E}$, s_∞ be L, $r' + B_0$ be A, and r' be B, we complete the proof of Theorem 3.1. □

4. Ending lamination theorem for topologically tame Kleinian groups

In this section, we shall state and prove our rigidity theorem for topologically tame Kleinian groups.

Theorem 4.1. *Let Γ_1 and Γ_2 be topologically tame Kleinian groups. Let C_1, C_2 be topological cores of $\mathbf{H}^3/G_1, \mathbf{H}^3/G_2$ respectively, each component of whose complement is homeomorphic to the product of a closed surface and an open interval. Suppose that there is a lower bound $\delta_0 > 0$ for the injectivity radii at all points of $\mathbf{H}^3/\Gamma_1$ and $\mathbf{H}^3/\Gamma_2$. Suppose moreover that there is a homeomorphism $h : \mathbf{H}^3/\Gamma_1 \to \mathbf{H}^3/\Gamma_2$ which maps C_1 to C_2, and an ending lamination on a boundary component S of C_1 for the end facing S to an ending lamination on $h(S)$ for the end facing it, and that the same holds for h^{-1}. Then h is homotopic to a liftable quasi-isometry. Consequently, there exists a quasi-conformal homeomorphism $\omega : S^2 \to S^2$ such that $\omega\gamma\omega^{-1} = f_\#(\gamma)$ for any $\gamma \in \Gamma_1$, where we identify Γ_j with $\pi_1(\mathbf{H}^3/\Gamma_j)$ for $j = 1, 2$.*

Remark 2. We can rephrase Theorem 4.1 above into the form of Theorem A in Minsky [20] using Sullivan's theorem in [30]: i.e., if f preserves "the end invariants" (that is, the conformal structures at infinity for geometrically finite ends and ending laminations for geometrically infinite ends), then f is isotopic to an isometry.

We shall prove Theorem 4.1 by the same way as in the proof of Theorem A in Minsky [20], constructing a model manifold for both $\mathbf{H}^3/\Gamma_1$ and $\mathbf{H}^3/\Gamma_2$ and a liftable quasi-isometry from the model. As we shall apply Theorem 3.1 to Γ_1 and Γ_2, we shall use the symbol G to denote a topologically tame Kleinian group such that the injectivity radii at all points of $M = \mathbf{H}^3/G$ are bounded below by δ_0, and construct a model manifold for M which depends only on the topological type of the core, conformal structures at infinity and ending laminations.

Before the construction, we need to prove some lemmata on pleated surfaces in $\mathbf{H}^3/G$ which are contained in a neighbourhood of a geometrically infinite end. These will be used to prove the existence of a liftable quasi-isometry from the model.

Lemma 4.2. *Suppose that a geometrically infinite end e of M faces a compressible boundary component S of a topological core C. There exist a neighbourhood E of e, and a constant $B > 0$ as follows. Let $g : (S, \tau) \to E$ be a pleated surface. Then there exists a geodesic triangulation of (S, τ) with only one vertex such that each loop formed by an edge has length at most B and represents a non-trivial primitive element of $\pi_1(M)$.*

Proof. Let $g : (S, \tau) \to \mathbf{H}^3/G$ be a pleated surface which can be lifted to a pleated surface $\tilde{g}$ to $\tilde{M}$ homotopic to ι. Then the lift $\tilde{g}$ is incompressible; hence the injectivity radii at points on (S, τ) are bounded by ϵ_0, which is a lower bound for the injectivity radii at points in $\tilde{M}$. It follows that there is a compact set K in the moduli space of S such that if we forget the marking of (S, τ), then it is contained in K. In other words, there is a compact set K' in $\mathcal{T}(S)$ such that for any pleated surface $g : (S, \tau) \to \mathbf{H}^3/G$ as above, there exists a hyperbolic structure τ' in K' which is isometric to τ after forgetting the markings. It is easy to see that there exists a constant B and, for any hyperbolic structure $\sigma \in K'$, there exists a geodesic triangulation of (S, σ) with only one vertex such that the loop formed by each edge has length at most B. This induces such a triangulation of (S, τ) for a pleated surface $g : (S, \tau) \to \mathbf{H}^3/G$ as above.

Next we shall prove that each loop formed by an edge of the triangulation represents a non-trivial element of $\pi_1(M)$ if we choose a sufficiently small neighbourhood E of e and assume that $g(S) \subset E$. By Lemma 7.1 in Canary [3], for a loop γ formed by an edge of the triangulation, if the loop $g(\gamma)$ is null-homotopic in M and $g(S)$ is far from the branching locus, then $g(\gamma)$ bounds a disc whose diameter is proportional to the length of γ. Since the bounded disc must intersect the branching locus as g can be lifted to an incompressible surface, this implies that the length of γ is bounded below by a constant proportional to the distance from $g(\gamma)$ to the branching locus. Because the length of γ is bounded above by B, by assuming that $g(S)$ is sufficiently far out to the end, this cannot happen.

Next suppose that γ_i does not represent a primitive element of $\pi_1(M)$. Regard S as a boundary component of the topological core C and γ_i as a simple closed curve lying on ∂C. Then by using the annulus theorem, we can prove that there exists an essential embedded annulus A in C whose boundary components are parallel to γ_i. (See Lemma 2.1 in Ohshika [25] for the proof.) Let D be a compression disc for S. (Such a disc exists because we assumed that S is compressible.) Since the free homotopy class of γ_i can be assumed to be contained in the Masur domain, the annulus A must intersect D essentially. By the usual cut-and-paste techniques, we can find a boundary-compression disc for the annulus A, which means that either A is inessential or there is a compression disc disjoint from γ_i. This is a contradiction. □

Now following Minsky [20], we shall define three "distance" functions. Let E be the component of the complement of C facing S. Let $g : S \to E$, $h : S \to E$ be two maps homotopic to the inclusion of S in $E \cup S$. Let $d_M(g,h)$ be the distance between $g(S)$ and $h(S)$ in M. Let $D_M(g,h)$ be the minimum for the lengths of the longest trajectories of homotopies between g and h in E, i.e.,

$$D_M(g,h) = \min_{\{H:\text{homotopy from } g \text{ to } h \text{ in } E\}} \max_{p \in S}\{\text{length}(H(p,t)) | t \in I\}.$$

Finally, we define

$$\tilde{D}_M(g,h) = \inf_{\phi}\{D_M(g,h') | h' = h \circ \phi\},$$

where ϕ ranges over all automorphisms of S isotopic to the identity.

Then we can prove the following analogue of Lemma 4.2 in [20] by the same argument as its proof, by using our Lemma 4.2 above when S is compressible, and the proof of Lemma 4.2 in [20] itself by taking a covering of M associated to $\pi_1(S)$ when S is incompressible.

Lemma 4.3. *Let E be a neighbourhood of a given geometrically infinite end e of M facing a boundary component S of a topological core C. For any $\kappa > 0, \delta > 0$, there exists a constant B as follows. Let $f : (S,\sigma) \to E$ and $g : (S,\tau) \to E$ be pleated surfaces homotopic in E and suppose that there exists a liftable (κ,δ)-quasi-isometry $\phi : (S,\sigma) \to (S,\tau)$ homotopic to the identity. Then there exists a homotopy $H : S \times I \to E$ such that $H(\ ,0) = f, H(\ ,1) = g \circ \phi$ and for each $x \in S$, $H(x,t)(t \in I)$ is a geodesic segment with length at most B. In particular, $\tilde{D}_M(f,g) \leq B$.*

Outline of Proof. Here we shall explain an outline of the proof of Lemma 4.2 in Minsky [20], and how our Lemma 4.2 is used to adapt his argument to our case. Take a loop β formed by an edge in the geodesic triangulation on (S,σ) as in

Lemma 4.2, where we regard our $f:(S,\sigma)\to\tilde{M}$ as $g:(S,\tau)\to\tilde{M}$ in Lemma 4.2. Let β^* be the closed geodesic homotopic to $f(\beta)$, and β' the geodesic arc homotopic to $\phi(\beta)$ preserving the basepoint. A homotopy $G: S^1\times I\to\tilde{M}$ from $f|\beta$ to $g|\beta'$ passing through β^* can be constructed so that the length of the trajectory of the basepoint (i.e., length$(G(\{v\}\times I))$ for the basepoint v) is bounded above by a constant depending only on $\chi(S)$ and the lower bound ϵ_0 of the injectivity radii. To bound the trajectory of the basepoint with respect to H, it is necessary to measure the homotopical difference between $H|\beta\times I$ and G. Consider a map T from a torus T^2 which is obtained by composing $H|\beta\times I$ and G^{-1}. Since $f(\beta)$ represents a primitive element of $\pi_1(\tilde{M})$ as asserted in Lemma 4.2, we can see that $T_\#(\pi_1(T^2))$ is generated by $[f(\beta)]$; in particular, the difference between the trajectory of the basepoint with respect to H and that with respect to G is homotopic to a power of $[f(\beta)]$. Since the trajectory with respect to G intersects β^* and has length bounded by a constant depending only on $\chi(S)$ and ϵ_0, it follows from elementary hyperbolic geometry that the trajectory with respect to H is also bounded. Applying this argument to each edge loop of the geodesic triangulation on (S,σ), the proof of Lemma 4.3 is completed. □

Furthermore we can prove the following lemma, which is an adaptation of Corollary 4.6 in [20] to our situation.

Lemma 4.4. *Let E be a neighbourhood of a given geometrically infinite end e of M which is apart from $\mathcal{N}(\gamma)$ at a distance greater than D_2 in Lemma 3.10, and can be lifted homeomorphically to $\tilde{M}$. Then for any $d_1>0$ there exists $d_2>0$, depending only on d_1, M and $\chi(S)$, such that if $g_1:(S,\sigma)\to E$ and $g_2:(S,\tau)\to E$ are pleated surfaces homotopic to the inclusion of S and $d_M(g_1,g_2)<d_1$, then $\tilde{D}_M(g_1,g_2)<d_2$.*

Proof. The pleated surfaces g_1 and g_2 can be lifted homeomorphically to $\tilde{g}_1,\tilde{g}_2$ in a neighbourhood $\tilde{E}$ of $\tilde{e}$, which is a lift of the end e to $\tilde{M}$. Then by Lemma 3.10, for our d_1, there exists b such that if $d_{\tilde{M}}(\tilde{g}_1,\tilde{g}_2)=d_M(g_1,g_2)<d_1$, then $d_T(\sigma,\tau)\le b$. It is easy to see that there exist a constant κ depending on b and a κ-bi-Lipschitz map from (S,σ) to (S,τ) isotopic to the identity. Thus our lemma follows from Lemma 4.3. □

We can choose a neighbourhood E in Lemma 4.4 so that there should be a bound K such that for any point x in E there exists a pleated surface homotopic to the inclusion of S within distance K by Corollary 3.6. Then we can apply the argument of the proofs of Lemmata 4.7, 4.8 in [20], and we get the following lemma.

Lemma 4.5. *We can choose a neighbourhood E of a given geometrically infinite end e as follows. Let $g_1:(S,\sigma)\to E$ and $g_2:(S,\tau)\to E$ be pleated surfaces into E. Then there exist constants a,b such that*

$$a^{-1}d_M(g_1,g_2)-b\le\tilde{D}_M(g_1,g_2)\le ad_M(g_1,g_2)+b$$

and

$$a^{-1}d_T(\sigma,\tau)-b\le\tilde{D}_M(g_1,g_2)\le ad_T(\sigma,\tau)+b.$$

Proof of Theorem 4.1. Now, let us construct a model manifold N in the following way, which is the same as that in Minsky [20]. Since we assumed that G is topologically tame, by Theorem 3.1, we can choose a topological core C of $\mathbf{H}^3/G$ such that each component E of $M-C$ containing a geometrically infinite end is a neighbourhood of the end which satisfies the three conditions of Theorem 3.1, and each component of ∂C facing a geometrically finite end is the boundary component of the convex core facing that end. Let C' be a compact Riemannian manifold homeomorphic to C. At each component S of $\partial C'$, we attach $S \times [-1, \infty)$ to C' by identifying S with $S \times \{-1\}$, and construct a model manifold N by giving a Riemannian metric as follows.

First suppose that S corresponds to a boundary component of C facing a geometrically finite end. Take a hyperbolic metric $d\rho^2$ which is conformal to the quotient of a component of the region of discontinuity by its stabilizer, corresponding to the end facing S. We give a Riemannian metric on $S \times [0, \infty)$ by the equality

$$ds^2 = dt^2 + \cosh(2t) d\rho^2,$$

where $t \in [0, \infty)$.

Next suppose that S corresponds to a boundary component of C facing a geometrically infinite end. By the construction of C, the component E of $M - C$ containing the geometrically infinite end satisfies the three conditions of Theorem 3.1. Hence there is a Teichmüller geodesic ray L as in Theorem 3.1. For the Teichmüller geodesic ray $L : [0, \infty) \to \mathcal{T}(S)$, there are two measured foliations Φ_h and Φ_v such that $e^{-r}\Phi_h$ and $e^r\Phi_v$ determine a Euclidean structure with isolated singularities which is conformal to $L(r)$ for each $r \in [0, \infty)$. We can denote the Euclidean structure at $L(r)$ by

$$ds_S^2[r] = e^{2r} dx^2 + e^{-2r} dy^2,$$

where $\frac{\partial}{\partial x}$ is tangent to $e^{-r}\Phi_h$ and $\frac{\partial}{\partial y}$ is tangent to $e^r\Phi_v$. Then we define a metric on $S \times [0, \infty)$ by

$$ds^2(p, r) = ds_S^2[r](p) + dr^2.$$

We give a smooth metric m on N so that $m|C'$ should be the given metric on C' and so that for each boundary component S of C', $m|S \times [0, \infty)$ should be the metric defined above, by filling the gap $S \times (-1, 0)$ smoothly for each S.

Let us construct a map $f : N \to M$ which we shall prove to be a liftable quasi-isometry. First, since there is a canonical homeomorphism $id : C' \to C$, we define $f|C'$ to be id. Second, let S be a boundary component of C' such that $id(S)$ faces a geometrically finite end. Let E be the component of the complement of C in M facing $id(S)$. For any point $x \in E$, let $r(x)$ be the point on $id(S)$ nearest to x. As is shown in Morgan [17], this map r is a deformation retraction. Let E' be the set consisting of points apart from $id(S)$ at least distance 1. Let us define a map $R : E' \to S \times [1, \infty)$ by $R(x) = (r(x), d(x, r(x)))$. We define $f|S \times [1, \infty)$ to be the inverse of R. As is shown in Minsky [20] using the main result of Epstein-Marden [8], if we endow $S \times [0, \infty)$ with the metric $dt^2 + \cosh(2t)d\rho^2$, where $d\rho^2$ is the hyperbolic metric on S induced from the conformal structure on the quotient of a component of the region of discontinuity invariant under the subgroup corresponding to $\pi_1(S)$, then the map R is a liftable quasi-isometry; hence so is $f|S \times [1, \infty) \to E'$.

Third, suppose that $id(S)$ faces a geometrically infinite end. Then there is a Teichmüller geodesic ray $L : [0, \infty) \to \mathcal{T}(S)$ satisfying conditions (2) and (3) in Theorem 3.1. Let $g_n : (S, \rho_n) \to E$ be a pleated surface such that $d_T(\rho_n, L(n)) \leq B$. Recall that $L(r)$ is represented by the metric $ds_S^2(r)$, which induces a hyperbolic metric $\sigma(r)$. We isotope ρ_n so that $(S, \sigma(n))$ and (S, ρ_n) should be B'-bi-Lipschitz, where B' is a constant depending only on B, and compose the map isotopic to the identity, which isotoped ρ_n, to g_n from the right. (Such a constant B' exists by Lemma 2.5 in [20].) We then define that $f(p,n) = g_n(p)$. There are universal constants κ, δ such that the identity map from (S, ρ_n) to (S, ρ_{n+1}) is a liftable (κ, δ)-quasi-isometry, as is shown in Lemma 5.6 in [20]. Then by Lemma 4.3, there exists a homotopy $H_n : S \times [0,1] \to M$ whose trajectories have lengths bounded above by a constant depending only on κ, δ. We define $f(p,r)$ to be $H_{[r]}(p, r-[r])$ for $(p,r) \in S \times [0,\infty)$. By defining $f|S \times [-1,1]$ in the case when S faces a geometrically finite end, and $f|S \times [-1,0]$ otherwise so that f should be smooth, we get a map $f : N \to M$.

We can prove that f is a liftable quasi-isometry by exactly the same argument as in Minsky [20]. To prove Theorem 4.1, we again use the same argument as in [20], which is as follows. We have only to consider the case when for each geometrically finite end e of $\mathbf{H}^3/\Gamma_1$, the conformal structures at infinity of e and $f(e)$ are conformal by a homeomorphism homotopic to the restriction of f, because we can take a quasi-conformal conjugate of Γ_2 which has the property above by Ahlfors-Bers theory. Let N_1 be the model manifold for $\mathbf{H}^3/\Gamma_1$, and N_2 be that for $\mathbf{H}^3/\Gamma_2$. Let C'_j be a copy of the core contained in N_j for $j = 1, 2$. Recall that we defined ending laminations so that the image of an ending lamination by an auto-homeomorphism of the core acting on the fundamental group by an inner automorphism is again an ending lamination. Thus if $[\lambda]$ is the projective lamination defining the Teichmüller geodesic ray which was used to construct an end of N_1 corresponding to the end e, and $[\lambda']$ is that for an end of N_2 corresponding to the end $h(e)$, it may be possible that $h(|\lambda|)$ and $|\lambda'|$ differ by an automorphism of C_2 acting on $\pi_1(C_2)$ trivially. However, by composing this automorphism to h, we can assume that $h(|\lambda|) = |\lambda'|$. Furthermore, as in Minsky [20], we can prove that λ and λ' are uniquely ergodic by using the result of Masur [16]; hence by replacing one of them by its scalar multiple without changing the model manifolds, we can assume that $h(\lambda) = \lambda'$. Thus we can assume that the model manifolds of $\mathbf{H}^3/\Gamma_1$ and $\mathbf{H}^3/\Gamma_2$ are the same one, N. Thus there are liftable quasi-isometries $f_1 : N \to \mathbf{H}^3/\Gamma_1$ and $f_2 : N \to \mathbf{H}^3/\Gamma_2$. Then $f_2^{-1} \circ f_1$, where f_2^{-1} denotes the quasi-inverse of f_2, lifts to a quasi-isometry F from $\mathbf{H}^3$ to $\mathbf{H}^3$ which conjugates Γ_1 to Γ_2. By Theorem 3.5 in [20], whose proof can be found for example in Thurston [28] or Ghys-de la Harpe [9], F can be extended to a quasi-conformal homeomorphism from S^2_∞ to S^2_∞ which conjugates Γ_1 to Γ_2. □

5. Topologically conjugate Kleinian groups

In this section, as an application of Theorem 4.1, we shall prove the following theorem on topologically conjugate Kleinian groups.

Theorem 5.1. *Let Γ_1 be a topologically tame Kleinian group, and Γ_2 be another Kleinian group. In the case when Γ_1 is a free group, we assume that Γ_2 is also topologically tame. Suppose that there exists a homeomorphism $f : S^2_\infty \to S^2_\infty$ such that $f\Gamma_1 f^{-1} = \Gamma_2$ as groups of conformal automorphisms of S^2_∞. Suppose moreover that the injectivity radii at all points of $\mathbf{H}^3/\Gamma_1$ and $\mathbf{H}^3/\Gamma_2$ are bounded below by a positive constant ϵ_0. Then there exists a quasi-conformal homeomorphism $\omega : S^2_\infty \to S^2_\infty$ such that $\omega\gamma\omega^{-1} = f\gamma f^{-1}$ for any $\gamma \in \Gamma_1$.*

Remark 3. In the above theorem, although we did not assume that Γ_2 is topologically tame except in the case of free groups, it will be shown that Γ_2 is also topologically tame while we prove the theorem. Furthermore, we shall not use the assumption on the injectivity radii to prove that Γ_2 is topologically tame. Hence, except for the case of free groups, we shall prove that any Kleinian group topologically conjugate to a topologically tame group is also topologically tame.

Before starting to prove Theorem 5.1, we shall prove two lemmata whose idea is basically due to Cannon-Thurston [6].

Lemma 5.2. *Let G be a topologically tame Kleinian group. Let C be a topological core of $\mathbf{H}^3/G$ and S a boundary component of C facing a geometrically infinite end. Let $\{\gamma_i\}$ be a sequence of simple closed curves on S whose projective classes converge to an ending lamination. Let γ_i^* be the closed geodesic homotopic to γ_i in $\mathbf{H}^3/G$. Then for any geodesic $\tilde{\gamma}_i^*$ in $\mathbf{H}^3$ which is a lift of γ_i^*, the endpoints of $\tilde{\gamma}_i^*$ converge to a common point after taking a subsequence.*

Proof. By taking a subsequence, we can assume that the endpoints of $\tilde{\gamma}_i^*$ converge in S^2_∞ to some points x, y respectively. We shall use the same subscript i for this convergent subsequence. We have to prove that $x = y$. Suppose, on the contrary, that $x \neq y$. Then there is a geodesic $\tilde{\gamma}_\infty^*$ in $\mathbf{H}^3$ whose endpoints are x and y. We can easily see that $\tilde{\gamma}_i^*$ converges to $\tilde{\gamma}_\infty^*$ uniformly in any compact set in $\mathbf{H}^3$. Take a compact set $K \subset \mathbf{H}^3$ which intersects $\tilde{\gamma}_\infty^*$. Then for sufficiently large i, the geodesic $\tilde{\gamma}_i^*$ also intersects K. It follows that γ_i^* intersects the projection of K into $\mathbf{H}^3/G$, which is compact, for sufficiently large i. This contradicts the fact that $\{\gamma_i^*\}$ tends to an end, which follows from the assumption that $[\gamma_i]$ converges to an ending lamination. (See § 4 in Ohshika [24] for an explanation of this fact.) $\square$

The converse of the lemma above holds under the assumption that $\{[\gamma_i]\}$ converges inside the projective Masur domain.

Lemma 5.3. *Let G be a Kleinian group and C be a core of $\mathbf{H}^3/G$. Let S be a boundary component of C. Let $\{\gamma_i\}$ be a sequence of closed curves in $\mathbf{H}^3/G$. Let γ_i^* be the closed geodesic homotopic to γ_i in $\mathbf{H}^3/G$. Suppose that for any choice of lifts $\tilde{\gamma}_i^*$ of γ_i^*, the endpoints of $\tilde{\gamma}_i^*$ converge to a single point after taking a subsequence. Then γ_i^* tends to an end of $\mathbf{H}^3/G$. In particular, if γ_i is a simple closed curve on S and the projective class of γ_i converges to a projective lamination λ in the projectivized Masur domain of S, then λ is an ending lamination.*

Proof. We shall prove this lemma by contradiction. Suppose that $\{\gamma_i^*\}$ does not tend to an end. Then, as is shown in § 4 of [24], which is a generalization of the argument of Bonahon [2], there is a simply connected compact set K in $\mathbf{H}^3/G$ which intersects γ_i^* for sufficiently large i. Lift K to a compact set $\tilde{K}$ in $\mathbf{H}^3$. Then some lift $\tilde{\gamma}_i^*$ intersects $\tilde{K}$ for sufficiently large i. It follows that for any subsequence of $\{\tilde{\gamma}_i^*\}$, the endpoints cannot converge to a common point. This contradicts the assumption. The last sentence of the lemma follows easily from the definition of ending lamination. $\square$

Proof of Theorem 5.1. We shall prove that there is a homeomorphism $h : \mathbf{H}^3/G_1 \to \mathbf{H}^3/G_2$ which induces an isomorphism $h_\# : G_1 \to G_2$ such that $h_\#(\gamma) = f\gamma f^{-1}$ for every $\gamma \in G_1$. The main step of the proof is Lemma 5.4 below, in which we shall prove the existence of a homomorphism between cores.

When G_1 is a free group, G_2 is also topologically tame by assumption. Hence $\mathbf{H}^3/G_1$ and $\mathbf{H}^3/G_2$ are homeomorphic open handle bodies, and a homotopy equivalence is homotopic to a homeomorphism for such manifolds. Thus we can assume that G_1 is not a free group from now on (until we finish proving the existence of a homeomorphism h). Let C_1 and C_2 be topological core of $\mathbf{H}^3/G_1$ and $\mathbf{H}^3/G_2$ respectively, and suppose that each component of $\mathbf{H}^3/G_1 - C_1$ is homeomorphic to the product of a closed surface and an open interval. We shall prove that under this assumption, there is a homeomorphism $\overline{h} : C_1 \to C_2$ which induces the same isomorphism between the fundamental groups as the conjugation by f, and that G_2 is also topologically tame.

Lemma 5.4. *In the above situation, there is a homeomorphism $\overline{h} : C_1 \to C_2$ which induces the same isomorphism between the fundamental groups as the conjugation by f. Furthermore, G_2 is topologically tame.*

Proof. As $fG_1f^{-1} = G_2$, there is a homotopy equivalence $k : C_1 \to C_2$ such that $k_\#(\gamma) = f\gamma f^{-1}$ for all $\gamma \in G_1$. By Waldhausen's theorem in [32], for proving that k is homotopic to a homeomorphism, it is sufficient to prove that for each boundary component S of C_1, the restriction $k|S$ is homotopic into ∂C_2. As is shown in Ohshika [26], this is easy to prove when S faces a geometrically finite end, by considering components of the region of discontinuity and their stabilizers.

Suppose that S faces a geometrically infinite end e. If S is incompressible, we can prove that $k|S$ is homotopic into ∂C_2 by the argument in [26], using Thurston's covering theorem.

Now suppose moreover that S is compressible. Let G_1^S be a subgroup of G_1 associated with $\iota_\#(\pi_1(S))$, and let G_2^S be $fG_1^Sf^{-1}$. Let C_1^S and C_2^S be cores of $\mathbf{H}^3/G_1^S$ and $\mathbf{H}^3/G_2^S$ respectively. The topological conjugation by f induces a homotopy equivalence $k' : C_1^S \to C_2^S$.

Claim 3. The homotopy equivalence k' is homotopic to a homeomorphism from C_1^S to C_2^S.

Proof. Since S is a boundary component of C_1, the core C_1^S is a compression body whose exterior boundary $\tilde{S}$ is homeomorphic to S and projected to a surface homotopic to S by the covering projection from $\mathbf{H}^3/G_1^S$ to $\mathbf{H}^3/G_1$. On the other hand, since C_2^S is homotopy equivalent to a compression body C_1^S, even when C_2^S is not a compression body, C_2^S is obtained by attaching 1-handles to $\Sigma \times I$ for a possibly disconnected orientable closed surface Σ. The surface Σ corresponds homeomorphically to the union of the interior boundary components via a map which is a homotopical inverse of the homotopy equivalence k'. (Refer to § 3 in Ohshika [24] for a proof of these facts.) If for each component Σ_j of Σ, either $\Sigma_j \times \{0\}$ or $\Sigma_j \times \{1\}$ is disjoint from the attached 1-handles, then C_2^S is a compression body and k' is homotopic to a homeomorphism.

Suppose that this is not the case. Then there is a component Σ_j of Σ such that each of the boundary components of $\Sigma_j \times I$ ($\Sigma_j \times \{0\}$ or $\Sigma_j \times \{1\}$) has a 1-handle attached to it. Let $\overline{\Sigma}_j$ be an interior boundary component of C_1^S which is mapped to a surface homotopic to Σ_j by k'. (Such a boundary component exists, as noted in the last paragraph.) Then $k'|\overline{\Sigma}_j$ is not homotopic to a boundary component of C_2^S.

Let e_j be an end of $\mathbf{H}^3/G_1^S$ facing $\overline{\Sigma}_j$. Then since the end e_j faces the incompressible boundary component $\overline{\Sigma}_j$ of C_1^S, by the same argument as before, which appeared in [26], we can prove that $k'|\overline{\Sigma}_j$ is homotopic to a boundary component of C_2^S no matter whether or not e_j is geometrically finite. This is a contradiction, and we have completed the proof of the claim. □

Conclusion of the Proof of Lemma 5.4. Since C_1 and C_2 are cores, the homotopy equivalence $k : C_1 \to C_2$ can be extended to a homotopy equivalence $\tilde{k} : \mathbf{H}^3/G_1 \to \mathbf{H}^3/G_2$. Let γ_i^* be the closed geodesic homotopic to γ_i in $\mathbf{H}^3/G_1$. Then by Lemma 5.2, the endpoints of any lift $\tilde{\gamma}_i^*$ of γ_i^* to $\mathbf{H}^3$ converge to a common point in S^2_∞ as $i \to \infty$, after taking a subsequence. Let δ_i^* be the closed geodesic homotopic to $\tilde{k}(\gamma_i)$ in $\mathbf{H}^3/G_2$. Let $\tilde{\delta}_i^*$ be a lift of δ_i^* to $\mathbf{H}^3$. Then for the endpoints x_i, y_i of $\tilde{\delta}_i^*$, there is a lift $\tilde{\gamma}_i^*$ of γ_i^* whose endpoints are mapped to x_i and y_i by f. Because f is a homeomorphism, the endpoints x_i, y_i also converge to a common point as $i \to \infty$, after taking a subsequence. The closed geodesics δ_i^* can be lifted to closed geodesics $\hat{\delta}_i^*$ in $\mathbf{H}^3/G_2^S$, since δ^* is freely homotopic to an element contained in $k_\# \circ \iota_\#(\pi_1(S))$. By Lemma 5.3, $\{\hat{\delta}_i^*\}$ tends to an end of $\mathbf{H}^3/G_2^S$.

By the claim above, we know that C_2^S is a compression body homeomorphic to C_1^S and k' is homotopic to a homeomorphism. Let S_2 be the exterior boundary component of C_2^S. Then there is a homeomorphism $h_S : S \to S_2$ which is homotopic in $\mathbf{H}^3/G_2^S$ to the lift of $k|S : S \to \mathbf{H}^3/G_2$ to $\mathbf{H}^3/G_2^S$. Therefore $\hat{\delta}_i^*$ is homotopic to a simple closed curve $\hat{\delta}_i$ on S_2 such that $[\hat{\delta}_i]$ converges in $\mathcal{PL}(S_2)$ to $[h_S(\lambda)]$, which is contained in the projectivized Masur domain. Since $\hat{\delta}_i$ is contained in the Masur domain for sufficiently large i, it cannot be homotopic to a closed curve on a boundary component of C_2^S other than S_2. This can be easily proved by reductio ad absurdum using the cut-and-paste technique for the intersection of an annulus, which realizes a homotopy between $\hat{\delta}_i$ and a closed curve on the interior boundary, and a compression disc for S_2. (Refer to Otal [27] and Canary [3].) Hence $\hat{\delta}_i^*$ tends to the end facing S_2. Thus the end e_2^S facing S_2 has a neighbourhood homeomorphic to $S_2 \times \mathbf{R}$, as was proved in Proposition 4.12 in [24], and $h_S(\lambda)$ is an ending lamination for the end e_2^S of $\mathbf{H}^3/G_2^S$ facing S_2. Furthermore, as S_2 is the unique compressible component of ∂C_2^S, this implies that G_2^S is topologically tame.

Let $p_2 : \mathbf{H}^3/G_2^S \to \mathbf{H}^3/G_2$ be the covering associated with the inclusion. By the covering theorem proved in Canary [4], the restriction of p_2 to a small neighbourhood of the end e_2^S is a finite-sheeted covering of its image, and in particular, $p_2|S_2$ is homotopic to a finite-sheeted covering of a boundary component S_2' of C_2. Since $k|S$ is homotopic to $p_2 \circ h_S$ as maps to $\mathbf{H}^3/G_2$, hence also in C_2, the restriction $k|S$ is homotopic in C_2 to a finite-sheeted covering of the boundary component S_2' of C_2, and the end facing that boundary component S_2' has a neighbourhood homeomorphic to $S_2' \times \mathbf{R}$.

Thus by applying the argument above for each compressible boundary component of C_1 which faces a geometrically infinite end, and using Waldhausen's theorem, we have shown that there is a homeomorphism $\overline{h} : C_1 \to C_2$ such that $\overline{h}_\#(\gamma) = f\gamma f^{-1}$ for all $\gamma \in \pi_1(C_1) = G_1$. Hence, in particular, each boundary component of C_2 is an image of that of C_1 by $\overline{h}$, and by the argument above, we can see that each end of $\mathbf{H}^3/G_2$ has a neighbourhood homeomorphic to the product of a closed surface and $\mathbf{R}$, which implies that G_2 is topologically tame. □

Conclusion of the Proof of Theorem 5.1. By the uniqueness of the topological core proved in McCullough-Miller-Swarup [21], there is an auto-homeomorphism $s : \mathbf{H}^3/G_2 \to \mathbf{H}^3/G_2$ acting on $\pi_1(\mathbf{H}^3/G_2)$ by an inner automorphism which maps C_2 to a topological core C_2' such that each component of $\mathbf{H}^3/G_2 - C_2'$ is homeomorphic to the product of a closed surface and an open interval. (The latter core exists since G_2 is topologically tame.) Thus we can extend the homeomorphism $s \circ \overline{h}$ to a homeomorphism $h : \mathbf{H}^3/G_1 \to \mathbf{H}^3/G_2$ such that $h_\#(\gamma) = f\gamma f^{-1}$ for any $\gamma \in G_1$.

In the following, we shall denote C_2' above by C_2 and $s \circ \overline{h}$ by $\overline{h}$ for simplicity. Then each component of $\mathbf{H}^3 - C_2$ is homeomorphic to the product of a closed surface and an open interval, and $\overline{h} : C_1 \to C_2$ is a homeomorphism such that $\overline{h}_\#(\gamma) = f\gamma f^{-1}$ for all $\gamma \in \pi_1(C_1) \cong G_1$. (At this point, we allow G_1 to be a free group. The topological cores as above exist even when G_1 is a free group.) Now we shall show that $\overline{h}$ and $\overline{h}^{-1}$ map ending laminations to ending laminations for corresponding ends. We assume that G_1 is not isomorphic to a closed surface group from now on. Our theorem in the case of closed surface groups was already proved by Minsky [20].

Let λ be an ending lamination for an end e, which lies on the boundary component S of C_1 facing the end e. Then there is a sequence of simple closed curves $\{\gamma_i\}$ such that $[\gamma_i]$ converges to $[\lambda]$ in $\mathcal{PL}(S)$ and the closed geodesic γ_i^* homotopic to γ_i tends to e. Let $\delta_i = \overline{h}(\gamma_i)$, and let δ_i^* be the closed geodesic homotopic to δ_i in $\mathbf{H}^3/G_2$. Then $[\delta_i]$ converges to $[\overline{h}(\lambda)]$ in $\mathcal{PL}(\overline{h}(S))$. As before, by mapping lifts of γ_i^* in $\mathbf{H}^3$ by f, considering the endpoints, and applying Lemmata 5.2 and 5.3, we can see that $\overline{h}(\lambda)$ is an ending lamination. Since C_2 is not homeomorphic to $S \times I$, and λ is maximal, δ_i cannot be homotoped into a component of ∂C_2 other than $\overline{h}(S)$. (This can be proved as before by considering the intersection of an annulus and a compression disc when S is compressible. Otherwise, we use the annulus theorem of Jaco-Shalen-Johannson (refer to Jaco [11]).) Hence $f(\lambda)$ is an ending lamination for the end facing $\overline{h}(S)$. Thus by Theorem 4.1, we get a quasi-conformal homeomorphism ω as in the statement, and the proof is completed. □

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